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ON THE SYNTHESIS OF CLEAVAGE CHROMOSOMES

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The question which has prompted the present inquiry is, "How is it possible for the chromosomes in rapidly segmenting eggs to synthesize so quickly the new material needed in the reduplication process prior to each cell division?" While it is not to be supposed that a full answer can be given to this question at present, nevertheless, if we consider modern ideas of protein and chromosome structure and behavior and some other facts scattered in the cytological literature of the past, there emerges a surprisingly simple and illuminating hint as to one of the ways this may be accomplished. Here it is proposed to draw attention to the pertinent evidence and some of its rather obvious or possible implications.

Chemists have shown that proteins are composed of long chain-molecules, each link of the chain being a chemical unit such as an amino-acid residue, a pyrimidine ring or the like, united with other structural units by rather simple bonds. Thus, two amino-acids may react to form a molecule of water and the "residues," as they are called, are linked by a "peptid" bond, and when many units are involved we have a "polypeptid" chain. When new molecules of a complex protein are formed the lattice of the old molecule is supposed to act as a form or mold, each unit of which, such as an amino-acid radical, attracts, or somehow receives from the surrounding medium, a unit like itself and in this way the new molecule is organized and built up. Such a method of formation allows us to understand how the high degree of specificity of proteins (and presumably of genes) is maintained. Viewed in this light, the reduplication 5 to 10 or more times of the same chromosome during 24 hours in segmenting eggs would involve the utilization of relatively large amounts of the constituent units of the nucleoproteins of the chromosomes and would be much more easily understood were we to assume that the egg cytoplasm is extremely rich in the

structural units out of which nucleoproteins are made rather than to assume that these needed materials are synthesized anew, at some phase of the division cycle, out of more or less undifferentiated food material.

The early cytological literature is replete with descriptions of the various cellular mechanisms involved in the formation of yolk within the animal egg. In general there are two common methods, either through the agency of nurse-cells, or by the activity of the ovum itself. The cytological picture of the egg, in these two instances, is very different.

When the eggs grow at the expense of nurse-cells, the latter increase rapidly both in the size of the nucleus and the cytosome, and there is a very marked growth in the amount of chromatin as is shown by Feulgen's nucleal reaction. In the ovary of *Drosophila melanogaster*, for example,[4] the nurse-cell nuclei increase in diameter from 5 μ to 40 μ or more. This means that the volume of the original nuclei has been increased about 512 times. Judging from the figures of other investigators of nurse-cells (e.g., Gross,[1] or Jörgensen,[2]) this size increase is in no way unusual. As the growth of the ovum nears completion, the nurse-cell contents are usually absorbed by the egg either through a direct engulfing, as in many Diptera, or more indirectly by absorption through the egg wall, during which process the nurse-cell nucleus loses its chromaticity and dwindles in size along with the cell cytoplasm until the nurse-cell remains are shrivelled bits of débris which completely disappear. Thus directly or indirectly the nurse-cell contents are absorbed and large quantities of nuclear material enter the cytoplasm of the egg. An essentially similar condition obtains in the egg cells of many plants. Generally speaking, when nurse-cells function in yolk formation the egg nucleus remains small and stains very lightly.

When the formation of yolk is carried out by the egg itself, the egg nucleus, or germinal vesicle, invariably shows a great increase in size and this is accompanied by the formation of numerous nucleoli and often elaborate chromatic structures such as the so-called "lampbrush" chromosomes. After the formation of yolk is completed, the amount of chromatin which enters the first polar spindle is only a very small portion of the total amount visible when the germinal vesicle is at the height of its activity. Most of this chromatin, or its derivatives, is discarded into the egg cytoplasm, when the germinal vesicle breaks down, and is absorbed. Thus as regards nucleoproteins the situation is essentially similar to that in eggs nourished by nurse-cells.

In some animals both ovum and nurse-cells may function simultaneously or, as in the case of aphids, the summer eggs may be formed chiefly through the activity of a large germinal vesicle, while the yolk of the slow-growing winter eggs is due to the activity of nurse-cells. In the latter, the egg nucleus remains quite small and relatively achromatic.

The point of central interest for us is that by the cytological mechanisms

employed in the growth of ova prior to fertilization, the cytoplasm of the egg is the recipient of large amounts of chromatin, or its derivatives, either through an engulfing or an indirect absorption of nurse-cells and their nuclei, or from the breakdown of the germinal vesicle.

The facts cited have been known to cytologists for a long time but it is only recently that we have gained a clear insight into the nature of the changes in the chromosomes which go hand in hand with any great increase in nuclear size.

In larval tissues of insects, growth is often accomplished by an increase in nuclear and cell size rather than by cell division. Accompanying this increase in nuclear size Geitler[3] has shown that there is a series of intra-nuclear chromosome divisions so that ultimately these larval somatic cells reach a high degree of polyploidy. For example, the large lobed nuclei in the salivary gland of *Gerris* have either 1024 or 2048 complete sets of chromosomes, oenocytes are 128-ploid and nuclei in the septum walls of the testis are commonly 16-ploid. In Diptera, the salivary gland chromosomes exhibit a special type of polyploidy in which the chromatids (after somatic synapsis) remain closely associated together.

Recently the writer and E. R. Reindorp[4] made a study of the nurse-cells in the ovary of *D. melanogaster* and we found very clear evidence for a series of intra-nuclear chromosome division cycles going hand in hand with an increase in the size of nurse-cell nuclei so that by the time a diameter of 40 μ is reached there are probably 512 haploid sets of chromosomes present in each nurse-cell. Since there are 15 nurse-cells associated with each egg, in the fruit fly, and these are all eventually absorbed by the egg cytoplasm, it is obvious that prior to fertilization and cleavage the ovum receives the materials of thousands of homologous chromosomes.

No one has as yet made a study of the growth of germinal vesicles to determine if intra-nuclear division cycles occur here, but our modern cytological outlook compels us to infer that the great increase in the chromaticity of these nuclei is due to some sort of reduplication of the constituent chromosomes and there is much evidence already which suggests that "lampbrush" chromosomes may not be simple pachytene chromosomes, but chromosome aggregates. In the first place, if one goes back to Rückert's original description of how lampbrush chromosomes are formed in, for example, *Pristiurus*,[5] we are confronted with the fact that a typical pachytene thread, 10 μ long and 0.5 μ in diameter grows into a structure 100 μ long and 10 μ in diameter. During this growth Rückert says that the "Mikrosomen" (chromomeres) put out side branches which lie at right angles to the long axis; these side branches are chromomeric in structure and, were one of them to lie separately, it would be considered a single chromosome. These side branches would account for the increase in the breadth of the chromosomes but not their ten-fold increase in length. A

second and very suggestive fact is that in the nurse-cells of *D. melanogaster*, at a definite point in the intra-nuclear division cycle of large nuclei, the homologous chromatids, derived from the repeated divisions of the progeny of a single chromosome, form a hairy-caterpillar-like aggregate which is strikingly similar in form to the lampbrush chromosomes. There is a tendency for the separate chromatids to lie parallel to each other in these aggregates, and were one to transplant such an aggregate to a vertebrate egg undoubtedly it would be called a "lampbrush" chromosome. We are now making at my laboratory a study of the way typical lampbrush chromosomes are formed, but in the meantime it seems reasonably safe to conclude, in the light of all the evidence, that the growth of the germinal vesicle in eggs is accompanied by some sort of reduplication of the constituent chromosomes. As I have pointed out, only a very small part of the chromatin in the germinal vesicle enters the first polar spindle, and the remains of the germinal vesicle are absorbed by the egg plasm. This means that the material from thousands of chromosomes is set free in the cytoplasm, and just as in eggs with the nurse-cell mechanism, this would be available for use by the cleavage chromosomes.

There arises now the question: What happens to the chromosomes (or their derivatives) when they enter the egg cytoplasm, either by the nurse-cell or the germinal vesicle route? That they do not persist as visibly organized structures is well established cytologically and since the most diverse types of eggs have been tested by Feulgen's nucleal stain, with negative results, it appears that nucleic acid, as such, does not persist. On the other hand, there is reason to believe that the constituent proteins and nucleoproteins of the chromosomes do persist in a partially broken down form, because a number of chemical analyses have shown, in diverse eggs, the presence of large amounts of substances closely related to nucleic acid. Thus, in the mature egg of *D. melanogaster* Caspersson[6] has shown by his ultra-violet photo-electric method, that in the egg cytoplasm there is a high concentration of substances containing the pyrimidine ring. And Caspersson and Schultz[7] report that in the eggs of XX females there is appreciably less of this pyrimidine ring bearing material than in XXY eggs. This is in line with what we would expect in view of the fact that this substance is derived from the breakdown of nurse-cell chromosomes. Also in a number of marine and other eggs, many of which are of the germinal vesicle type, Brachet[8] and other workers have found a high concentration of nucleotides which we can understand in view of the large amount of nuclear material which is set free in the egg cytoplasm by the breakdown of the germinal vesicle.

The evidence, then, indicates that in the cytoplasm of all eggs there are the products of thousands of maternal chromosomes. Just in what form the constituent proteins and nucleoproteins exist is a matter for the bio-

chemist to determine. In the meantime, it seems reasonable to conclude that the rapid building up of the cleavage chromosomes is possible in the segmenting egg because the synthesis is more in the nature of a reassembling of already existing materials, such as nucleotides, etc., under the guidance of the active chromosomes, rather than an actual synthesis of the building blocks from relatively simple substances.

The presence of materials derived from a very large number of maternal chromosomes and genes in the cytoplasm of eggs not only allows us to understand the rapid reduplication of the cleavage chromosomes but also gives us a simple explanation for certain types of cytoplasmic or matroclinus inheritance. We commonly think of genes as forming specific substances which react with other cellular constituents to produce, in the end, phenotypic expressions. Furthermore, Ephrussi[9] has shown that substances produced early in ontogeny may persist and affect structures developed in late larval life. This being true we might anticipate that the presence of large amounts of material derived from the maternal chromosomes and genes might sometimes affect the F_1 phenotype irrespective of the genotype of the latter. Many different cases of matroclinus inheritance, especially those which deal with larval characters, seem best understood in this light, and adult characters may occasionally be affected. Thus Noujdin[10] finds that the presence of a Y chromosome in the female fruit fly tends to suppress mottling and that all the progeny of such XXY females also show the same suppression no matter what their chromosome and genetic constitution may be. Since all of the eggs of an XXY female receive from their nurse-cells thousands of Y chromosomes which enter the egg cytoplasm we may assume that products of these Y's persist and either quantitatively or qualitatively function to suppress the mottling. Since, however, the eggs of the F_1 females receive from their nurse-cells the nuclear products of their own genotype, the matroclinus suppression of mottling would persist only through the one generation.

Attention must also be given to the possible influence which the maternal chromosome products in the egg cytoplasm may play in development in general and especially in so-called parthenogenic merogony. The fact that Mrs. E. B. Harvey[11] has succeeded in stimulating enucleated egg fragments to develop in some instances to a morula stage would seem to minimize the importance of the chromosomes and genes in early development. Such a conclusion does not necessarily follow because it is quite possible that in the absence of a normal nuclear set-up, the genes, and possibly the division centers, brought into the egg by the nurse-cell or germinal vesicle route, may function in this type of abortive development.

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[2] M. Jörgensen, *Arch. Zellforsch.*, **10**, 1 (1913).
[3] L. Geitler, *Chromosoma*, **1**, 1 (1939).

[4] T. S. Painter, and E. R. Reindorp, *Chromosoma*, **1** (1939).
[5] J. Rückert, *Anat. Anz.*, **7**, 107 (1892).
[6] T. Caspersson, *Sk. Arch. Physiol.*, sup. to 73 (1936) and unpublished data.
[7] T. Caspersson, and Jack Schultz, *Nature*, **142**, 294 (1938).
[8] J. Brachet, *Arch. Biol.*, **48**, 529 (1937).
[9] B. Ephrüssi, *Amer. Nat.*, **72**, 5 (1938).
[10] N. J. Noujdin, *Nature*, **137**, 319 (1936).
[11] E. B. Harvey, *Biol. Bull.*, **71**, 101 (1936).

A NEW INHERITED CHARACTER IN MAN

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Observations on more than 280 human subjects show the existence of two fairly distinct classes with respect to the ability to turn up the lateral edges of the tongue. In typical positive cases the edges can be rolled together over a considerable portion of the distal area of the tongue, while the organ is slightly protruded. In negative cases there is no turning up of the edges at all. A few intermediates have been encountered; and in numerous cases the ability, at first absent, has been acquired by practice. This latter phenomenon is most frequent in children, only one clear case having been found in an adult—and here prolonged efforts were necessary, whereas in children a few hours are sometimes enough. One man reports that he learned the trick as a child, but now has forgotten it and can no longer do it. It should be added that some children, like most negative adults, appear to be unable to learn. In the data that follow, all cases where the ability was at first absent are entered as negative.

Another complication encountered is that a few children are unwilling to show whether they possess the ability or not. In one of these cases the child later said that this unwillingness was due to embarrassment because the ability was absent. The few (two) remaining such individuals have been entered in the tables as negatives.

The ability evidently has no relation to sex, as is shown by table 1:

TABLE 1

	POSITIVE	NEGATIVE	TOTAL	PER CENT NEGATIVE
Female	88	43	131	32.8 ± 4.1*
Male	95	56	151	37.1 ± 3.9*
Total	183	99	282	35.1 ± 2.8*

* Standard error.

Studies of families indicate that the ability is inherited, as shown by tables 2 and 3.

TABLE 2

MOTHER	FATHER	NUMBER OF FAMILIES	OFFSPRING POSITIVE	OFFSPRING NEGATIVE	TOTAL
Positive	Positive	18	28	5 (1)	33
Positive	Negative	11	16	11 (1)	27
Negative	Positive	14	17	11 (2)	28
Negative	Negative	4	4	9	13
Positive	Unknown	10	14	3	17
Negative	Unknown	2	4	0	4
Unknown	Positive	3	2	2 (1)	4
Unknown	Negative	1	0	1	1

(Numbers in parentheses indicate individuals known to have acquired the ability.)

TABLE 3

NUMBER IN FRATERNITY	NUMBER OF FAMILIES WITH NUMBER OF NEGATIVES INDICATED 0	1	2	3	4	5	6
2	11	7	2				
3	3	4	3	1			
4	2	0	0	1	0		
5	2	0	0	0	0	0	
6	0	0	0	1	1	0	0

Table 4 indicates that random mating occurs:

TABLE 4

HUSBAND AND WIFE	NUMBER OF PAIRS	EXPECTED
Positive × positive	25	27
Positive × negative	31	28
Negative × negative	6	7

As will be seen from table 2, neither class breeds true. Positive × positive has given five negatives, negative × negative has given four positives. The first cases do not seem doubtful, and have come in five separate families. Two of them have each a negative grandparent, and a third has a negative half-brother. This third child learned the trick in one day. The four positives from two negative parents occurred in two families (two in each); both of these families are from fathers with slight speech defects, which suggests that the negative tests on the fathers may be dependent on some additional complicating factor.[1] It is possible, though not proved, that ability to turn up the edges of the tongue may be due to a single dominant gene, with the fairly frequent occurrence of additional complications.

Two pairs of identical twins have been tested. All four individuals (which are included as separate ones in table 3) are positive—a result in

agreement with the supposition of simple inheritance of the character, but needing to be checked by additional observations.

The individuals in table 1 belong to a wide variety of races, but are mostly Americans of mixed European ancestry. Both positives and negatives have been observed in the following groups though the numbers are too small to make the proportions significant: English, Russian, Russian Jewish, Dutch, Polish, Negro (presumably hybrids with whites), Japanese.

The data here recorded have been collected by many observers. Since the members of a given family are usually recorded by the same observer, it might be supposed that the correlation between relatives is a spurious one, due only to differences in classifying intermediates. This supposition is negatived by the absence of a correlation between husband and wife, who are also usually recorded by the same observer.

Another possible interpretation of the data is that there is no truly genetic element, the correlations being dependent on family habits or customs, or on imitation in some form. This supposition is not supported by two sets of data: the effect of the father and of the mother on the ability of their offspring is equal; and there is no indication of a striking difference in frequencies of the two classes in the various national and language groups included among those studied.

Summary.—The ability to turn up the edges of the tongue, present in about 65 per cent of the persons studied, is conditioned at least in part by heredity.

[1] It should be added, however, that other individuals with slight speech defects have been found to be positive.

INFLUENCE OF FEMALE STOCK ON THE FUNCTIONING OF SMALL POLLEN MALE GAMETES

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Communicated December 20, 1939

In the many cases of natural or induced heterozygous small pollen conditions in maize it is now considered axiomatic that the small pollen grains do not function in competition with the normal, large grains. This is true whether the production of small pollen grains is associated with a detectable cytological deficiency or occurs in a stock in which the chromosomes show no visible abnormality.

In the case of small pollen-1, sp_1, Mangelsdorf[1,2] found that less than one per cent of the sp_1 pollen grains effected fertilization in competition with

normal. He demonstrated, however, that the sp_1 grains are capable of germination and affecting fertilization when screened to eliminate competition with the normal large pollen grains. In our investigations on the small pollen condition we have found, on the whole, that less than one per cent of the sp_1 pollen grains function in competition with normal.[3]

Consequently, it was surprising to find, in 1938, results of some crosses of $su \times \frac{sp\ su}{+\ +}$ that could be explained only by assuming a considerable proportion of the sp_1 pollen grains had accomplished fertilization. The pollinations were made on two different sweet stocks, Purdue 39 and Connecticut 81. Four sets of paired pollinations were made on these two inbreds. In every case the Purdue 39 had a considerably higher percentage of sweet seed indicating that the sp_1 pollen grains were able to function better when applied to this stock. In one case there was 53 per cent of *su* kernels. Here there was no competetive effect between the + and sp_1 pollen grains. The total counts for the pollinations on the two inbreds were as follows:

	Su	*su*	TOTAL	PER CENT *su* (C. O. CLASS)
Purdue 39 $\times \frac{sp\ su}{+\ +}$	2714	1744	4458	39
Conn. 81 $\times \frac{sp\ su}{+\ +}$	7089	1478	8567	17

In each of the pollinations on these two inbreds there should have been only about 6 per cent of sweet seeds (the crossover ratio between *sp* and *su*) had there been no functioning of sp_1 pollen grains. Both per cents, 17 for *C*81 and 39 for *P*39, are much too high for the crossover ratio and indicate a functioning of the sp_1 pollen grains. These figures also show that Purdue 39 silks function as a more favorable host to the small pollen grains than the Connecticut 81, if our interpretation is correct that the higher percentages are due to the functioning of the small pollen grains.

To test this point, the four different classes of seeds from the two pollinations were planted in 1939 and pollen of the plants was examined to determine how many were segregating sp_1. The following results were obtained:

	POLLEN *sp*/+	POLLEN + +	TOTAL	PER CENT *sp*/+
*P*39 $\times \frac{sp\ su}{+\ +}$ *su* seed	447	68	515	87
*C*81 $\times$ " *su* seed	302	233	535	56
*P*39 $\times$ " *Su* seed	9	439	448	2.0
*C*81 $\times$ " *Su* seed	18	527	545	3.3

Thus we see the *Su* seeds produced a very small per cent of plants with segregating pollen. This was expected since *Su* was linked with *Sp* and the only segregating plants from these seeds would come from a crossover be-

tween these two loci, and a functioning of the *sp Su* pollen grain after the crossover. Hence the percentage obtained would never be more than the per cent of recombination. The counts are too small to say whether the percentage of 2.0 is significantly different from 3.3.

In the case of the *su* kernels more segregating plants were expected since this is the linkage class of *sp* and *su*. Also a higher percentage was expected from the cross using Purdue 39. The *su* seeds from this cross produced plants, 87 per cent of which had segregating pollen, while the *su* seeds of the *C*81 cross produced only 56 per cent of segregating plants. These figures verified the assumption that the excess percentages of *su* (C. O. class) seeds in the original cross were really functional small pollen and did not represent an increase in crossing-over. It is possible to obtain the true recombination per cent in each cross by multiplying the original value by the per cent of non-segregating plants found (the true crossovers). When this is done the crossover per cent becomes 5.1 for the *P*39 crosses and 7.5 for the *C*81 crosses. These fluctuations from the normal 6 per cent C. O. value are probably not significant.

The pollen examinations showed conclusively there was considerable functioning of sp_1 pollen in competition with normal. They also showed a greater functioning when applied to Purdue 39, than when applied to Connecticut 81. The nature of this difference is a matter of speculation. Do the silks of Purdue 39 afford a better environment for the germination of sp_1 pollen grains? If so, is it possible by using this inbred as a female stock to secure the functioning of other small pollen male gametes? Other inbreds might conceivably be as favorable or more favorable than Purdue 39.

No explanation of this condition is available at present.

[1] Mangelsdorf, P. C., *Proc. Nat. Acad. Sci.*, **17**, 698–700 (1931).
[2] Mangelsdorf, P. C., *Jour. Hered.*, **23**, 289–295 (1932).
[3] Singleton, W. R., *Proc. 6th Internat. Genetics Congress*, **2**, 182–184 (1931).

GALACTIC AND EXTRAGALACTIC STUDIES, V. THE PERIOD FREQUENCY OF CLASSICAL CEPHEIDS IN THE MAGELLANIC CLOUDS

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Read before the Academy, October 24, 1939

In the study of the characteristics of stellar variation, a well-known observational result is the scarcity of periods for classical Cepheids[1] in the interval from 1.0 to 2.5 days and the conspicuous maximum in the period-frequency curve between four and five days. The intrinsic luminosity, the mass, mean density, temperature, diameter, color, spectrum, and all variations of the five-day Cepheid are naturally accepted as the typical characteristics for theoretical studies. There has been some question as to the reality of secondary maxima in the period-frequency curve at ten and sixteen days; but no doubt has arisen about the principal maximum, which as shown in figure 1 occurs at about 4.75 days, with the curve falling steeply on both sides, approaching minima near two and nine days.

In the present communication it will be shown that in the Small Magellanic Cloud, where we are much more free from the evils of selection than we are in the galactic system, the maximum is in the vicinity of two days. The long-accepted "infrequency" between 1.5 and 2.5 days has disappeared. It is probable that our sampling of classical Cepheids around the sun, although it involves more than three hundred stars, has been insufficient to yield full information on this useful characteristic of Cepheid variation—the distribution of periods; or, alternatively, we might conclude that, improbable as it seems, the galactic system and the Magellanic Clouds differ fundamentally in this one respect—the period distribution of their Cepheids.

Since the new study of the Magellanic Clouds also shows that the distribution of periods depends significantly on the distribution of stellar mass, it might be more correct to say that in the Small Magellanic Cloud as a whole the period frequency of classical Cepheids differs from that in our part of the galactic system either because of a different average star density or because selection has given an untrue picture of the phenomenon for galactic Cepheids.

1. *Period Frequency in the Galactic System.*—From the *Vierteljahrsschrift* Catalogue for 1939, the distribution of the periods of the 288 classical Cepheids with periods less than twenty days is given in table 1 and shown as a frequency graph in figure 1, where the plotted dots indicate numbers for each half-day interval and the open circles represent running means of three. There are nearly forty Cepheids now known with periods greater

than twenty days, but the distribution of their periods is here ignored as irrelevant. The minima in the period-frequency curve at 1.5 and at 9.0 days, noted by the Gaposchkins[2] and others, are shown in this plot. Most of the stars with periods less than 2.5 days are discoveries of recent years when the surveys have gone to fainter stars in Milky Way fields.

2. *First Comparison with the Small Magellanic Cloud.*—The period frequency of galactic Cepheids may be next compared with the published results for the Small Magellanic Cloud from which a somewhat comparable frequency curve is obtained. Omitting the periods of 22 outlying Cepheids, published in *Harvard Annals*, **90,** No. 4 (1934) and *Harvard Circular* 374 (1932), we obtain the results as given in the first line of table 2 and in figure 2. The maximum of this frequency curve also comes at 4.75 days, or when smoothed at 4.25 days. It is noted, however, that there are here relatively more of the shorter periods, and when we examine the results in *Harvard Circular* 374, where there are many periods less than three days, considerable doubt arises as to the similarity of the frequencies in the Small Cloud and the galactic system.

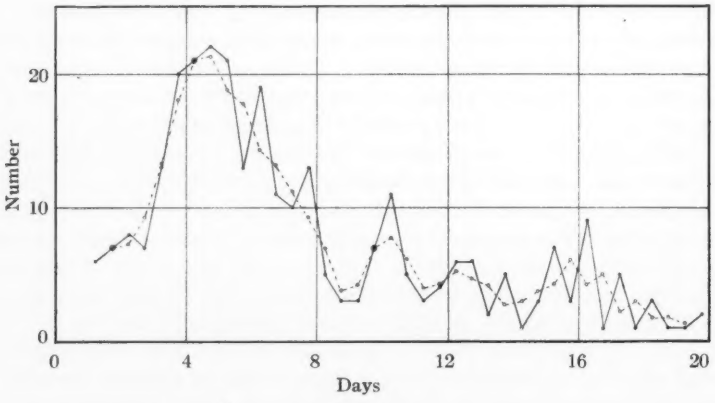

FIGURE 1

Distribution of periods for 288 galactic Cepheids. Ordinates are numbers of variables in half-day intervals of the period (abscissae). Broken lines with open circles represent running means of three points.

3. *A New Investigation of the Small Cloud Variables.*—In order to examine the phenomenon further, additional series of plates have been obtained with the Bruce refractor at the Boyden Station. During the past fifteen months approximately 14,000 observations have been made; a sixth of the known variables are now measured. Many of the periods and light curves previously published have been re-examined. With one excep-

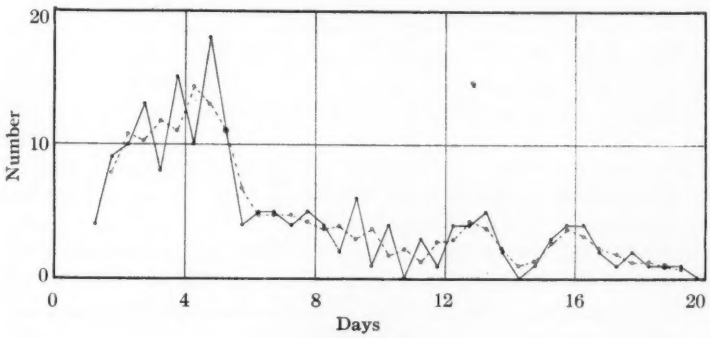

FIGURE 2

Distribution of periods in main body of Small Magellanic Cloud. Coördinates as in figure 1.

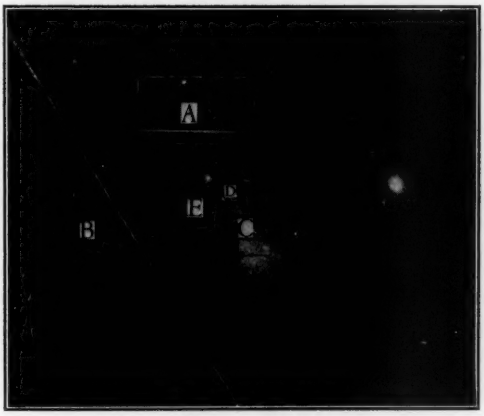

FIGURE 3

The Small Magellanic Cloud, with special regions marked (see table 2 and figures 4 and 5).

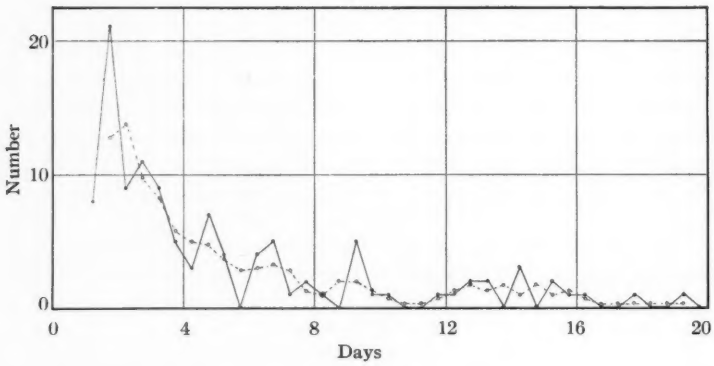

FIGURE 4

Distribution of periods for the fully investigated regions of the Small Magellanic Cloud. Coördinates as in figure 1.

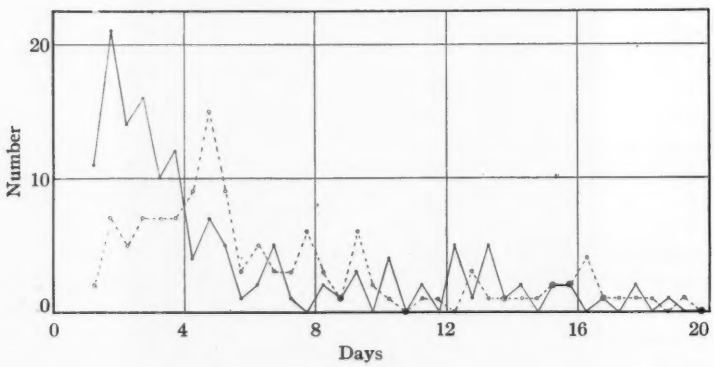

FIGURE 5

Distribution of periods in the "inner" and "outer" regions of the Small Magellanic Cloud, the broken line with circles representing the former. Coördinates as in figure 1.

tion, H. V. 216, the periods obtained by Miss Sawyer for the region near the globular cluster NGC 362 at the northern edge of the Cloud are verified.[3] This northern area has been enlarged and the Cepheids measured in that region more than doubled in number. New investigations in other parts of the Cloud were also undertaken when it became clear that not only should the period-frequency relation be revised but that the possible dependence of period length on star density must also be examined. The eighty new periods determined in the course of this work will be published elsewhere.

In order to examine in detail the distribution of periods throughout the Cloud, we have made table 2, in which the first column indicates various regions in the Cloud, and the tabulated quantities are the numbers of variables in half-day intervals of period, as indicated at the top of successive columns. The regions are marked also on the photograph reproduced in

TABLE 1

FREQUENCY OF PERIODS FOR GALACTIC CEPHEIDS (1 TO 20 DAYS)

PERIOD IN DAYS	NUMBER	PERIOD IN DAYS	NUMBER	PERIOD IN DAYS	NUMBER	PERIOD IN DAYS	NUMBER
1.0–1.5	6	6.0–6.5	19	11.0–11.5	3	16.0–16.5	9
1.5–2.0	7	6.5–7.0	11	11.5–12.0	4	16.5–17.0	1
2.0–2.5	8	7.0–7.5	10	12.0–12.5	6	17.0–17.5	5
2.5–3.0	7	7.5–8.0	13	12.5–13.0	6	17.5–18.0	1
3.0–3.5	13	8.0–8.5	5	13.0–13.5	2	18.0–18.5	3
3.5–4.0	20	8.5–9.0	3	13.5–14.0	5	18.5–19.0	1
4.0–4.5	21	9.0–9.5	3	14.0–14.5	1	19.0–19.5	1
4.5–5.0	22	9.5–10.0	7	14.5–15.0	3	19.5–20.0	2
5.0–5.5	21	10.0–10.5	11	15.0–15.5	7		
5.5–6.0	13	10.5–11.0	5	15.5–16.0	3		

figure 3. The total of 295 accepted classical Cepheid variables in the Small Cloud for which periods are now determined includes 72 for which periods are longer than twelve days. Since the distribution of these longer periods has no effect on the question of the most frequent period and the location of maxima and minima in the part of the frequency graph in which we are now interested, they have not been used in determining the median period, given for the various parts of the Cloud in the last column of table 2. But the accompanying figures include the data on periods up to twenty days, and for only thirty stars does the period exceed that value.

It should be emphasized that some of the data of table 2 are affected by selection. Until the present investigation was undertaken, the Cepheids chosen for period determination were generally those that could contribute effectively in the testing of the form of the period-luminosity relation, or those that for one reason or another were easiest to measure. As a result there has been a definite selection of the brighter stars and therefore of the

TABLE 2

DISTRIBUTION OF PERIODS IN THE SMALL MAGELLANIC CLOUD

	REGION IN CLOUD	PERIOD INTERVAL (IN DAYS) 1.0–1.5	1.5–2.0	2.0–2.5	2.5–3.0	3.0–3.5	3.5–4.0	4.0–4.5	4.5–5.0	5.0–5.5	5.5–6.0	6.0–6.5	6.5–7.0	7.0–7.5
	All Published*	4	9	10	13	8	15	10	18	11	4	5	5	4
A.	Northern Border	5	11	2	7	5	2	1	2	2	..	2	1	..
B.	Southeast Border	1	6	5	1	1	2	..	1	1	..	..	2	..
	Both Borders	6	17	7	8	6	4	1	3	3	..	2	3	..
C.	Nuclear	..	2	3	2	2	5	3	8	6	2	3	2	3
	(*a*) Core	..	..	..	1	..	1	..	2	1	..	1	1	1
D.	Intermediate	2	4	2	2	3	..	2	2	..	..	1	1	..
	(*a*) Dense	1	1	..	1	1	..	2	2	..	..	..	1	..
	(*b*) Sparse	1	3	2	1	2	..	..	..	..	..	1	..	..
E.	Inner	2	7	5	7	7	7	9	15	9	3	5	3	3
	Outer	10	21	14	16	10	12	4	7	5	1	2	5	1
	Total	12	28	19	23	17	19	13	22	14	4	7	8	4

	REGION IN CLOUD	7.5–8.0	8.0–8.5	8.5–9.0	9.0–9.5	9.5–10.0	10.0–10.5	10.5–11.0	11.0–11.5	11.5–12.0	>12	TOTAL NUMBER	MEDIAN PERIOD *d*
	All Published*	5	4	2	6	1	4	..	3	1	61	203	4.50
A.	Northern Border	..	..	..	1	..	..	..	..	..	1	42	2.77
B.	Southeast Border	..	1	..	1	..	1	..	..	..	13	36	2.48
	Both Borders	..	1	..	2	..	1	..	..	..	14	78	2.70
C.	Nuclear	2	3	1	4	1	..	..	1	..	20	73	5.12
	(*a*) Core	..	..	..	2	1	..	..	..	..	10	21	6.47
D.	Intermediate	2	..	..	1	..	..	..	..	1	2	25	3.17
	(*a*) Dense	1	..	..	1	..	..	..	..	1	1	13	4.70
	(*b*) Sparse	1	..	..	..	..	..	..	..	..	1	12	2.16
E.	Inner	6	3	1	6	2	1	..	1	1	35	138	4.85
	Outer	..	2	1	3	..	4	..	2	..	37	157	2.99
	Total	6	5	2	9	2	5	..	3	1	72	295	3.75

* Except those in *Harv. Circ.* 374, and *Harv. Ann.*, **90**, No. 4.

longer periods. Only in regions A, B, D and the "core" of C have all the variables been investigated, whatever the brightness and period.

The deliberate preference shown brighter stars in the determination of periods for the 177 variables represented in figure 2 has tended to force the resemblance of the Cloud's frequency curve to that of the galactic system (Fig. 1). If we make a frequency graph for only the four regions essentially free of selection, we obtain the result shown by figure 4—a surprisingly different picture than heretofore seen for the distribution of Cepheid periods. This might be taken as the true period distribution for Cepheids,

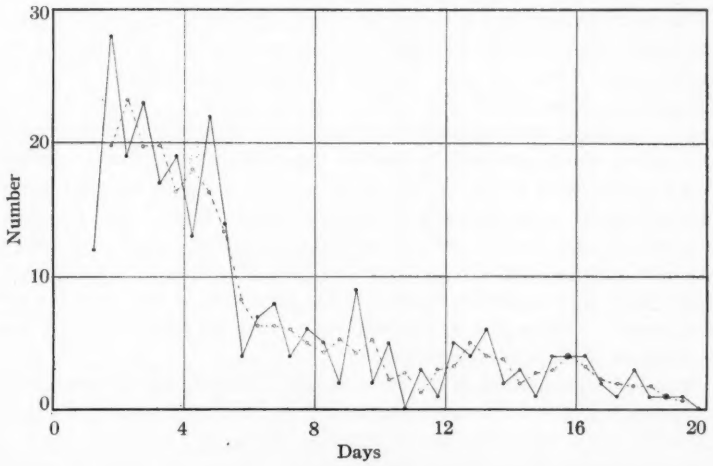

FIGURE 6

Distribution of the 265 periods now known for the Small Magellanic Cloud—the sum of the two curves in figure 5. The broken line with open circles is the running-mean curve. A remnant of the selection effect is seen in the secondary maximum at 4.75 days. Coördinates as in figure 1.

but the four regions involved lie mostly in areas of low star density and may not thoroughly represent the Cepheid population of the Cloud as a whole.

That the period frequency may depend sharply on the star density is shown, without much weight because of the small numbers involved, by the distribution of periods in region D—a small area chosen to cut across the border between high and low densities. When we divide this "intermediate" region into equal parts, as in table 2 and figure 3, we find that three-fourths of the periods in the sparse half are less than 3.5 days; nearly three-fourths of those in the denser half are greater than 3.5 days. The medians are 2.16 and 4.70 days, respectively.

Only a small section of the so-called nuclear region, *C*, was thoroughly investigated for the whole range of magnitude and period. In that "core" the median for stars with periods less than twelve days is 6.47 days, a value that may hold for the whole of the nucleus when all of its fainter classical Cepheids have been studied. It is noteworthy that about half the stars of the core have periods greater than twelve days, a ratio not approached in any other area studied. The considerable number of long-period Cepheids in region *B*, however, demands further investigation; these variables may be related to a peculiar extension of the Small Cloud toward the southeast, which we are now investigating as to population and structure.

The three bottom lines of table 2 are shown graphically in figures 5 and 6. For figure 5 all the 295 variables with known periods are divided into "inner" and "outer," in order to show with greater weight and clarity the dependence of the period distribution on star density. If the selection, mentioned above, were absent throughout the areas studied, the maxima of both curves would probably be shifted somewhat toward shorter periods.

The combination of the two curves of figure 5 is shown in figure 6, with the open circles again indicating running means of three. The maximum frequency in this "total" curve is at two days, instead of at 4.75 days as found for galactic Cepheids in figure 1 and for Magellanic Cepheids in the earlier study represented by figure 2. The minimum at nine days has now disappeared. The waves in the frequency curve for periods greater than ten days are perhaps not significant.

Probably it is too early in the study of period distribution to attempt the interpretation of the frequencies observed in the Small Cloud. It is not likely that any other significant selection effects are involved; and the plate material is now quite sufficient to provide ample tests of periods, magnitude sequences and background effects. We might suggest that equipartition of energy may account for the fact that the shorter period variables, with their lower luminosities and masses, appear preferentially in the regions of lower star density; but a simple calculation shows how relatively little the masses of two-day Cepheids differ from those of six-day period. Such fine sorting out on the basis of mass would probably demand an unacceptably long time scale and should require also other clear evidences of an approach toward a steady state.

An alternative hypothesis is that the distribution of matter and gravitational potential throughout the Cloud at some earlier time, or the distribution of the various chemical elements,[4] governed the masses and therefore the periods of the forming Cepheids; in other words, heavy stars formed in "heavy regions," or the chemical elements that favor the maintenance of long-period Cepheid variation may have been localized in the denser regions.

4. *Cepheids in the Large Cloud.*—It will be useful for the interpretation

of the characteristics of the period-frequency graphs in the Small Cloud to see what evidence there is elsewhere for a star-density effect on period distribution. Table 3, similar to table 2, shows what we can now say about the Large Magellanic Cloud. The variables already published[5] show the usual value of the median period, about 4.5 days. Selection has again entered through our special interest in improving the brighter end of the period-luminosity curve. For an area on the axis of the Cloud, and for another of lower star density which is, however, not so near the border as was attained for the Small Cloud, we have made a thorough study and find median periods of 5.24 and 4.36 days, respectively. The effect is in the same direction as for the Small Cloud, but no periods less than 2.5 days have as yet been found. On four special series of Bruce plates, which permit a better border patrol, the question will be further investigated. Mr. A. L. Jennings and Miss Frances W. Wright have assisted in the study of the Large Cloud.

5. *Further Consideration of Galactic Cepheids.*—We have carried through many studies of the bearing of location in the galactic system on the period frequency of classical Cepheids, with the following principal results:

a. The material for galactic Cepheids has been highly selected on the basis of apparent magnitude and convenience of the ob-

TABLE 3

DISTRIBUTION OF PERIODS IN THE LARGE MAGELLANIC CLOUD

REGION IN CLOUD	1.0–1.5	1.5–2.0	2.0–2.5	2.5–3.0	3.0–3.5	PERIOD INTERVAL (IN DAYS) 3.5–4.0	4.0–4.5	4.5–5.0	5.0–5.5	5.5–6.0	6.0–6.5	6.5–7.0	7.0–7.5
All Published*	..	..	..	8	8	12	10	10	2	2	3	5	3
Axis	..	..	..	..	3	2	..	2	2	2	..	2	..
Outer	..	..	..	1	2	3	2	2	1	..	..	1	..
Total	..	..	..	9	13	17	12	13	5	4	3	8	3

REGION IN CLOUD	7.5–8.0	8.0–8.5	8.5–9.0	9.0–9.5	9.5–10.0	10.0–10.5	10.5–11.0	11.0–11.5	11.5–12.0	>12	TOTAL NUMBER	MEDIAN PERIOD d
All Published*	1	2	..	1	1	1	1	2	1	33	106	4.42
Axis	1	1	1	..	..	..	..	..	..	2	18	5.24
Outer	1	..	..	..	1	..	..	..	1	..	15	4.36
Total	2	3	1	1	2	1	1	2	2	35	137	4.63

* This number includes three unpublished periods by Miss Sawyer and six by Miss McKibben.

server (latitude, seasons, equipment). Also the irregular distribution of space absorption and the irregular structure of surrounding star fields combine to make valid conclusions impossible.

b. The periods average shorter in higher latitudes, where the star density is certainly lower; but since the boundaries of the galactic system enter the problem, another kind of selection may be largely responsible for this result also.

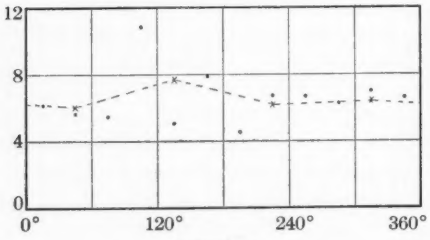

FIGURE 7

The relation of median period (ordinates) to galactic longitude (abscissae) for galactic Cepheid variables with periods less than twenty days. The crosses represent median periods for 90° intervals of longitude; the dots for 30° intervals.

c. Figure 7 shows that there is no appreciable dependence of period-length on galactic longitude in the material now available.[6] It is based on the following median values of galactic longitude and period (number of stars in last line) for all galactic Cepheids with median apparent magnitudes brighter than 10.0 and periods less than twenty days:

λ	15°	45°	75°	105°	135°	165°	195°	225°	255°	285°	315°	345°
P	6.1	6.6	5.4	10.8	5.0	7.9	4.5	6.7	6.7	6.3	7.0	6.6
No.	9	10	9	6	5	7	6	9	23	12	5	9

The Cepheids in the direction of the galactic center (longitude 325°) may on the average lie in regions of greater star density and mass than prevail in the anticenter direction, but we cannot be certain because of our present ignorance of the details of galactic structure.

d. It is easy to show that our records for galactic Cepheids are progressively less complete for the more distant stars, those of shorter period being preferentially omitted so long as we do not approach the boundaries too closely.[7] We conclude, therefore, that for a given volume of space in the galactic system the period-frequency maximum would shift toward shorter periods than suggested by figure 1. No matter how complete the survey, however, we may not be able to attain in this part of the galactic system results comparable to those for the Small Magellanic Cloud (figure 4 or 6), since it is possible that the star density near the Sun corresponds not to the average but to the densest part of the Small Cloud.

Summary. a. The periods and median magnitudes of eighty classical Cepheid variables in the Small Magellanic Cloud have been determined in

order to supplement earlier material for an investigation of the form of the period-frequency curve and the dependence of median period on position in the star cloud.

b. It is found that the maximum frequency at about 4.5 days for galactic Cepheids holds also in the Small and Large Clouds when a strong selection for the bright long-period Cepheids affects the material, but when the selection is eliminated the maximum of the frequency curve shifts to approximately two days, where heretofore a minimum of frequency had been supposed to occur.

c. Between the center and the borders of the Small Cloud the median frequency differs by two or three days (table 2), an indication that the Cepheids in the regions rich in stars are systematically brighter and more massive than in the outer parts of the Cloud.

d. Preliminary results for the Large Magellanic Cloud are presented. An investigation of the galactic Cepheids with respect to the dependence of period on position in the Milky Way gives inconclusive results, probably because of the many vitiating factors of selection in the discovery and analysis of galactic variable stars.

[1] "Classical" Cepheids are those with periods in excess of one day. For recent discussions of period distribution see, for example, Lundmark, K., *Vierteljahrsschrift d. Ast. Gesell.*, **68**, 377 ff. (1933); Campbell, L., *Pop. Ast.*, **47**, 216 (1939).

[2] Gaposchkin, C. P., and S., *Variable Stars*, Harvard Observatory Monograph No. 5, 162 (1938).

[3] But the uncertainties in range and period for H. V. 1869 lead us to omit it for the present.

[4] The possibility that the abundance of lithium, berylium, boron and deuterium may have a bearing on the form of the period-frequency curve of galactic Cepheids has been suggested by Gamow, *Nature*, **144**, 575 (1939).

[5] *Harv. Bull.* 905, 22 (1937).

[6] With less material the relation of period to longitude has been previously examined by Lundmark (see footnote 1) and earlier by Schilt, *Ap. J.*, **64**, 149 (1926), and *Ast. Jour.*, **38**, 197 (1928). In our tabulation and plot we omit variables with magnitudes fainter than 10.0 because of the proximity of the boundary of the galactic system in the anti-center direction and the consequent restriction on average period length—high luminosity stars in those longitudes being necessarily absent among Cepheids of faint apparent magnitude.

[7] Some of the effects of selection will be discussed and evaluated in a paper appearing elsewhere.

REMARKS ON ZWICKY'S PAPER "ON THE FORMATION OF CLUSTERS OF NEBULAE AND THE COSMOLOGICAL TIME SCALE"

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Communicated January 3, 1940

In a paper recently published in these PROCEEDINGS[1] Zwicky has marshalled arguments purporting to show that the existence of large clusters of nebulae is inconsistent with the cosmological short time scale demanded by the theory of the expanding universe. His main argument appears to be that the formation of clusters of nebulae requires multiple close encounters; an estimate of the interval between collisions leads him then to the conclusion that the time of formation of a large cluster out of a random distribution would be greater than 10^{18} years, a figure obviously so much greater than the age of expanding universe as to rule out the latter. He then examines the possibility that clusters might already have existed in the early stages of the expansion and dismisses it on the ground that, according to the investigations of Sinclair Smith, the velocity distribution in the central part of a cluster is the same as at a considerable distance from the center, and large radial velocities at the rim of a cluster are not observed, as would be expected if they had shared in the expansion of the universe. He finally suggests an experimental verification of the theory of the expanding universe which would consist in attempting to find out if remote clusters are more condensed than nearby ones.

While formulating these objections to the theory in question, Zwicky seems to have overlooked certain important results of Lemaitre on the theory of the formation of clusters of nebulae in an expanding universe. In a paper appearing almost six years ago in these PROCEEDINGS[2] he showed that in such a universe there may be locally collapsing regions and equilibrium regions, and went on to examine the hypothesis that the former are to be identified with the extragalactic nebulae and the latter with the clusters of nebulae. This hypothesis implies, first, that the mean density of all clusters should be the same, and second, that they may be of any shape, including the spherical. He then tested his first conclusion on eight clusters reported by Hubble and Humason and on twenty-five studied by Shapley, and found a very satisfactory agreement with his theoretical expectation. The suggestion that the collapsing regions are the extragalactic nebulae led, in Lemaitre's hands, to a rather satisfactory answer to the question of the origin of the energy involved in the creation of stars within a nebula.

The purpose of this note is to point out that, so long as Lemaitre's con-

clusions as outlined in his papers have not been disproved, it cannot be seriously maintained that the existence of clusters of nebulae in any way contradicts the theory of the expanding universe, or the cosmological time scale associated with it.

[1] F. Zwicky, these PROCEEDINGS, **25**, 604 (1939).

[2] G. Lemaitre, *Ibid.*, **20**, 12 (1934).

Elaborations and further tests of these ideas were presented by Lemaitre in a paper read at the University of Notre Dame's symposium on cosmic physics, held in April, 1938. This paper has not, to the writer's knowledge, ever been published.

THE OPACITY OF EXTENDED STELLAR ATMOSPHERES

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Communicated January 4, 1940

1. In a recent paper[1] the expanding atmosphere of P Cygni was treated as a layer of gas which is transparent for radiation except within the absorption lines. The equivalent widths of the absorption lines furnished a measure of N_2H—the number of atoms per cm.^{2} above the photosphere responsible for the lines. The intensities of the emission lines, on the other hand, furnished a measure of the quantity $(1/W)N_nH$, where W is the dilution factor and N_nH is the number of atoms per cm.^{2} above the photosphere, in the upper level of each emission line. Adopting a relation of the Boltzmann type between N_2 and N_n, we should, in principle, be able to derive W. The conclusion was that the emission lines are relatively too weak, and that, in consequence, the value of W was much larger than that derived from spectroscopic criteria, such as the relative intensities of lines arising from metastable and from normal levels.

2. The difficulty is removed if we abandon the assumption that the expanding shell in P Cygni is transparent for continuous radiation. There is, in fact, abundant evidence that the shell has a considerable optical thickness. If we compare P Cygni, 17 Leporis and HD 190073, we notice that these three stars show, with increasing clearness, the underlying spectrum of a stationary layer, which in HD 190073 and 17 Leporis gives rise to typical broad hydrogen absorption lines whose contours are produced by Stark effect. In 17 Leporis, moreover, we observe an undisplaced absorption line of Mg II 4481. There can be no doubt that the expanding shell of HD 190073 is almost perfectly transparent. That of 17 Leporis is already markedly affected by continuous absorption and emission in the shell. The broad hydrogen wings are too weak for class A_0, but the star cannot

be placed among the B's, or among the later A's, because we fail to observe undisplaced He lines and we also fail to observe strong undisplaced lines characteristic of the later subdivisions of class A. Nor can the underlying spectrum be classified as that of a normal giant, for in that case it would show a *strong* undisplaced line of Mg II 4481 and, in addition, strong undisplaced components of Fe II, Ti II, etc. Finally, in P Cygni there is no trace left of the undisplaced lines, or of the Stark wings of the hydrogen lines, and we must conclude that the optical thickness of the expanding shell is considerable.

A survey of the *Be* stars shows that, in general, when the outer shell gives rise to strong, narrow absorption lines of hydrogen, as in φ Persei, the normal stellar lines of He and the broad wings of H are always very weak. This phenomenon is quite conspicuous, but its recognition has been delayed by the well-known correlation between the widths of the emission lines and the rotational broadening of the He lines.[2] The latter tends to make the He lines appear rather inconspicuous. It requires a comparison of the He lines in such *Be* stars as φ Persei with normal stars having similar rotational velocities to ascertain the effect of the continuous absorption of the shell.

3. We may consider the atmosphere of a *Be* star to consist of: (1) a normal layer below a certain optical thickness, $\tau_1 = \int_0^{z_0} \kappa\rho\, dz$, where κ is the continuous absorption coefficient of the shell; and (2) of a tenuous shell above τ_1. A stratified model of a stellar atmosphere has been treated by Eddington[3] and by Swings and Chandrasekhar.[4] Adopting the case of pure scattering ($\epsilon = 0$, in Eddington's notation) and setting

λ_{Gr} = absorption coefficient of atoms within the line
κ_{Gr} = absorption coefficient in the continuous spectrum
$\eta = \dfrac{\lambda}{\kappa}$

we find for the residual intensity within the line

$$H'/H = 1 - \frac{\sqrt{3}\eta}{2\left\{\left[1+\eta+\frac{\sqrt{3(1+\eta)}}{2}\right]\sinh\sqrt{3}\,\tau_1 + \left[\sqrt{1+\eta}+\frac{\sqrt{3}(1+\eta)}{2}\right]\cosh\sqrt{3}\,\tau_1\right\}}.$$

Consider a point within the Stark-wing of a normal hydrogen line, as produced in the stationary layer below τ_1. Let the intensity of this point in a star not surrounded by a shell be $H'/H = 0.6$. Then, for $\tau > \tau_1$ we have $\eta = 2$. Above τ_1 we have, of course, $\eta = 0$. Numerical computations show that if $\tau_1 = 0.1$, then $H'/H = 0.66$. This is approximately correct for 17 Leporis. For P Cygni we must have $\tau_1 \gtrsim 0.5$.

These estimates rest solely upon the decrease of the intensities of absorp-

tion lines when seen through a shell. The question arises whether they are consistent with the known absorption coefficients and the known amounts of matter in the shells. Consider first the case of 17 Leporis. We have not yet been able to measure accurately the equivalent widths of the absorption lines. But since the hydrogen lines are similar to those of P Cygni, we may estimate $N_2H = 10^{15}$. In order to estimate the optical thickness, we must know the number of hydrogen atoms in the neutral state. Hence we apply the Boltzmann formula (and disregard possible departures from thermodynamic equilibrium):

$$\log \frac{N_2}{N} = -\chi_2 \frac{5040}{T} + \log \frac{g_2}{u}.$$

For 17 Leporis $T = 10{,}000°$. This gives, approximately,

$$N = 10^{20}.$$

The continuous absorption coefficient per gram of neutral hydrogen,[5] for $T = 10{,}000°$, and at λ 5000 is

$$\kappa_{gr} = 20.$$

Since $m_H = 1.7 \times 10^{-24}\, gr$, we find that the optical depth produced by 10^2 neutral hydrogen atoms at $T = 10{,}000°$ is

$$\tau_1 = 3 \times 10^{-3}.$$

We require $\tau_1 = 10^{-1}$. Hence, hydrogen is not sufficient to produce the required opacity, provided our estimate of N_2H is correct. The continuous absorption of the metals and of the free electrons may easily increase τ_1 to approximately 10^{-2}. The remaining discrepancy—by a factor of about 10—is not sufficiently serious to cause concern, and there is at present no need to search for other sources of continuous absorption.

These considerations cannot be applied directly to P Cygni. Here the optical thickness of the shell is so great that our observational value of NH does not, even approximately, represent the number of atoms above τ_1. Hence, we must proceed in a different manner. In the determination of N_2H we had assumed that the atmosphere is transparent and that re-emission can be neglected. Hence, we adjusted N_2H in such a manner that $e^{-N_2H\lambda_{al}}$ correctly represented the observed equivalent width. A weak line was used. Hence

$$e^{-N_2H\lambda_{al}} \approx 1 - N_2H\lambda_{al}.$$

The stratified atmosphere must now be considered to contain absorbing atoms only above τ_1. Therefore $\eta = \eta_0$ when $\tau < \tau_1$ and $\eta = 0$ when $\tau > \tau_1$. The continuous absorption coefficient κ is constant for all values of τ.

The corresponding solution of the problem of radiative transfer by Eddington[3] and by Swings and Chandrasekhar[6] gives[7]

$$H'/H = f(\eta, \tau_1) = 1 - \frac{\eta}{1+\eta} \cdot \frac{\frac{q}{\sqrt{3}} \sinh q\tau_1 + \cosh q\tau_1 - 1}{\left(\frac{2}{q} + \frac{q}{\sqrt{3}}\right) \sinh q\tau_1 + \left(1 + \frac{2}{\sqrt{3}}\right) \cosh q\tau_1},$$

where $q = \sqrt{3(1+\eta)}$.

Accordingly, we may write

$$NH\lambda_{at} = 1 - f(\eta, \tau_1).$$

But

$$\eta = \frac{\lambda_{gr}}{\kappa_{gr}} = \frac{\lambda_{at}\zeta}{\mu_{gr}\ m_H},$$

where ζ is the ratio of the atoms in the lower level of the line to all atoms of the gas and μ is the mean atomic weight of the unionized gaseous mixture.

We infer again from Unsöld's table[5] that for $T = 25{,}000°$ and $\lambda = 5000$, the continuous absorption coefficient per gram of neutral hydrogen is

$$\kappa_{gr} = 2 \times 10^5.$$

Let us consider a specific point on the contour of the absorption line produced by the shell, for which $\eta = 1$. Then

$$\lambda_{gr} = 2 \times 10^5.$$

If we are dealing with a shell consisting of pure hydrogen, then ζ is given by the Boltzmann factor, provided κ_{gr} is expressed per gram of neutral hydrogen:

$$\zeta = 10^{-2}.$$

Accordingly, $\lambda_{at} \approx 3 \times 10^{-17}$. For the number of atoms we found[8] $N_2H \approx 10^{15}$. The problem is, therefore, to adjust τ_1 in such a manner that the following expression is fulfilled

$$3 \times 10^{-2} = 1 - f(\eta, \tau_1).$$

By means of trials we find, approximately,

$$\tau_1 = 0.05.$$

This is again too small, by a factor of 10. But we have allowed only for absorption in hydrogen. Other sources of continuous absorption will increase this value, and will probably be sufficient to remove the discrepancy.

The absorption lines of the shell are of the same order of strength as the normal absorption lines of a giant star of similar ionization. Hence, if the

absorbing properties of matter are the same in a normal giant reversing layer and in a shell, it is obvious that the optical depth τ_1 will be roughly 0.5—this being Unsöld's value[9] for the wings of normal absorption lines. Hence the discrepancies found for 17 Leporis and P Cygni would, if they were real, merely suggest greater values of κ than those which we have used.

4. The recognition of the existence of appreciable continuous absorption in the shells of P Cygni and 17 Leporis suggests an explanation of the relative weakness of the emission lines, mentioned in section 1. It is clear that the absorption lines will be favored if the emission lines come only from that part of the shell for which the optical thickness is less than that of the fictitious photosphere ($\tau_1 < 1/3$, according to Milne).

Another interesting consequence of our results is their bearing upon Kosirev's theory[10] of reddening in extended atmospheres. We know now that in P Cygni the opacity of the expanding shell is considerable. From the fact[1] that the dilution factor is of the order of 0.1, we conclude that the total thickness of the shell may be equal to the radius of the star. For such a layer it is no longer permissible to neglect the curvature—and the problem of radiative transfer must be solved along the lines suggested by Chandrasekhar[11] and by Kosirev.[10] The effect of reddening follows from this treatment.

It is probable that the same explanation holds in the case of γ Cassiopeiae. According to the observations of Greaves and Martin[12] the color temperature of this star dropped from 16,100°K. in 1926.78 to 9200°K. in 1937.72. This change was accompanied by the outburst of a shell which produced strong absorption lines,[13] and which must have had a considerable optical thickness.

The problem of the absorption lines is not seriously affected by the curvature of the shell. Chandrasekhar has derived a simple formula[14] for an infinite atmosphere. The case of a stratified curved atmosphere has not been investigated, but it is clear that the change will be negligible.

5. The question may arise whether the broad lines in 17 Leporis and φ Persei really arise in the same star the outer shell of which produces the sharp lines. In the case of φ Persei this question has been answered by an observation of Hynek[15] who has found from measurements of line displacements that the broad hydrogen wings and the sharp cores come from the same source. Another question is whether the outer shell envelops all parts of the photosphere, or whether we are dealing here with a phenomenon similar to that of the solar prominences. It is conceivable that the sharp lines are formed in a large mass of separate prominences, and that we are able to see the photosphere through the open spaces between the individual prominences. Observations of the sharp and very strong line Ca K in 17 Leporis show no appreciable intensity in the center. Hence the continuous spectrum of the normal photosphere is not superposed over the

violet-displaced line of the expanding shell. We infer that the shell is essentially unbroken, and that the broad hydrogen wings are seen through the shell and not between separate prominences.

[1] O. Struve and F. E. Roach, *Ap. Jour.*, **90**, 727 (1939).
[2] *Ap Jour.*, **73**, 94 (1931).
[3] *M. N.*, **89**, 626 (1929); *The Internal Constitution of the Stars*, German Edition, pp. 424–425 (1928).
[4] *M. N.*, **97**, 33 (1936).
[5] A. Unsöld, *Die Physik der Sternatmosphären*, p. 121 (1938).
[6] *M. N.*, **97**, 32 (1936).
[7] The form of this equation is due to Mr. Ralph Williamson of the Yerkes Observatory.
[8] *Ap. Jour.*, **90**, 749 (1939).
[9] Op. cit., p. 247.
[10] *M. N.*, **94**, 430 (1934).
[11] *Ibid.*, **94**, 444 (1934); *Russian Astr. Jour.*, **6** (6), 8 (1934).
[12] *Ibid.*, **98**, 434 (1938).
[13] Baldwin, *Ap. Jour.*, **89**, 256 (Plate 17) (1939).
[14] *M. N.*, **94**, 456 (1934).
[15] Unpublished.

NORMAL ALGEBRAIC NUMBER FIELDS

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1. *Introduction.*—In this note we present a brief outline of investigations on the generalization of class field theory to arbitrary normal fields. A complete treatment of the results will be submitted for publication to the *Transactions of the American Mathematical Society*. In the sequel k denotes an algebraic number field of finite degree over the field of all rational numbers. Class field theory[1] deals with the description of abelian extensions over k in terms of classes of ideals in k. In other words, it is the analog of the theory of multiplicative functions on algebraic Riemann surfaces. It is possible to describe the class groups of the classical theory by means of the totality of all p-adic extensions k_p of k; p denoting the prime divisors[2] of k.

2. *Ideal Algebras.*—Let H_p denote a class of normal simple algebras with index m_p over k_p.[3] A set $\{H_p, \text{all } p\} = H$ shall be termed an *ideal algebra* if and only if $m_p = 1$ for all but a finite number of prime divisors. If $H^{(1)} = \{H_p^{(1)}\}$, $H^{(2)} = \{H_p^{(2)}\}$, then we define the product $H^{(1)} \times H^{(2)}$ as $\{H_p^{(1)} \times H_p^{(2)}\}$. This definition implies that all ideal algebras form a group H. Now let K be a normal extension of degree n over k with

Galois group $\Gamma = \{\sigma, \ldots, \tau\}$. Let $n_p = [K_P : k_p]$, where P is a typical prime divisor of p in k. We say that K *splits* an ideal algebra H if $n_p \equiv 0 \pmod{m_p}$ for all prime divisors p. Thus H contains as a subgroup the class group of all normal simple algebras S/k which are split by K.

In order to describe our generalization of class field theory we term "algebras" H or S *relatively prime* to a divisor $M = \Pi P^{\alpha}$ (in K) if no $P \frown k$ is a ramification divisor[4] of H or S, respectively. The group of all ideal algebras that are relatively prime to M can be called H_M. It contains the group S_M. Our investigations deal with the evaluation of the index

$$[H_M : S_M], \qquad M = \prod_P P^{\alpha(P)}$$

for arbitrary normal fields K/k and properly selected divisors M depending on the structure of K relative to k. If the field K is specialized to a cyclic field Z then the general index reduces to the index of the class group associated to Z in k by the class field theory.

3. *Theory of Primary Factor Sets.*—In order to actually evaluate the index $[H_M : S_M]$ we use a device first developed by Artin in some unpublished work.[5] Let $\Gamma_I = \{\Sigma_\sigma c_\sigma \sigma\}$ be the group ring of Γ with coefficients c_σ in the ring I of all integers. Suppose that P is an arbitrary prime divisor of K, Δ_P the associated decomposition group. The conjugates of P are obtained by applying to P substitutions $\lambda \in \Gamma$ which are representatives of Γ modulo Δ_P. We then consider the infinite cyclic group $\{P\}$ generated by P under the operators of Γ_I. A p-primary factor set $F\{P\}$ is a set of n^2 ideals $P^{a_{\sigma,\tau}}$ in $\{P\}$ with exponents $a_{\sigma,\tau}$ in Γ which satisfy the n^3 conditions

$$a_{\sigma,\tau} + a_{\sigma\tau,\rho} = \sigma a_{\tau,\rho} + a_{\sigma,\tau\rho}.$$

These associativity relations imply that, for suitable elements b_σ,

$$na_{\sigma,\tau} = b_\sigma + \sigma b_\tau - b_{\sigma\tau}, \text{ i.e.,}$$

$$(P^{a_{\sigma,\tau}})^n = P^{b_\sigma} P^{\sigma b_\tau} P^{-b_{\sigma\tau}},$$

a factor set of transformation quantities.[6] All factor sets of transformation quantities form a subgroup $T\{P\}$ of $F\{P\}$. The theory of group characters yields that the index of this subgroup is

$$[F\{P\} : T\{P\}] = [\Delta_P : \Delta_P'],$$

where Δ_P' denotes the commutator group of Δ_P. Thus, if $p = P \frown k$ is unramified in K we find

$$[\Delta_P : \Delta_P'] = f(p);$$

$f(p)$ denoting the relative residue class degree of a typical P dividing p. By the fundamental theorem of ideal theory we can collect factor sets $F\{P\}$ to factor sets $F\mathfrak{A}$ of arbitrary ideals $\mathfrak{A}$ in K.

A factor set $A_{\sigma,\tau}$ of numbers from K always determines a crossed product $(K/k, \Gamma, A_{\sigma,\tau})$ which is a normal simple algebra over k. In terms of these algebras we next define certain factor sets $(B_{\sigma,\tau})$ of principal ideals which correspond to the principal ideals generated by norm residues in the cyclic theory. The $B_{\sigma,\tau}$ are all those factor sets defined by the restriction that each $B_{\sigma,\tau}$ is prime to the divisor M, and the restriction on the corresponding crossed product that[7]

$$(K/k, \Gamma, B_{\sigma,\tau})_{\mathfrak{p}} \sim k_{\mathfrak{p}}, \text{ for all } \mathfrak{p} \text{ ramified in } K/k.$$

Let $F_M(A_M)$ denote the group of all factor sets $(B_{\sigma,\tau})$. Using the theory of rational algebras and the existence theorem for prime divisors with given Frobenius symbol we arrive at[8]

$$[H_M : S_M] = [F\mathfrak{A}_M : T\mathfrak{A}_M \cdot F_M(A_M)]$$

$$= \text{l. c. m. of all orders of elements } \sigma \in \Gamma.$$

The definition of the $B_{\sigma,\tau}$ also determines the divisor M for which we must consider H_M, S_M. We observe that these results yield ordinary class field theory only for cyclic extensions. Namely, if Γ is the fours group then $[H_M : S_M] = 2$, whereas the inversion theorem of class field theory yields the value 4 for the index of the class group.

4. *Determination of* $[F\mathfrak{A} : T\mathfrak{A}_M \cdot F_M(A_M)]$ *by Means of Index Reduction.*—Another way of evaluating the index in question is to generalize the proof of the classical inversion Theorem.[9] Lack of space does not permit us to sketch the important steps of our reduction. At various stages of the process we have to consider arbitrary normal extensions $K_P/k_{\mathfrak{p}}$. Let E denote the group of all units in K_P which are congruent to 1 modulo P. If ζ and η range over the decomposition group Δ_P, $E_{\zeta,\eta}$ will denote a corresponding factor set of units from the group E, while $F_\Delta E$ denotes the group of all such factor sets. We define the local conductor $C(P)$ of $K_P/k_{\mathfrak{p}}$ as the least power of P such that

$$E_{\zeta,\eta} \equiv 1 \pmod{C(P)}$$

implies

$$(K_P/k_{\mathfrak{p}}, \Delta_P, E_{\zeta,\eta}) \sim k_{\mathfrak{p}}.$$

It is readily seen that $C(P)$ is the unit divisor if $P/\mathfrak{p}$ is not ramified. If $P/\mathfrak{p}$ has no higher ramifications, then K_P is cyclic of degree prime to P over its unramified subfield and $C(P) = \mathfrak{p}$. The divisor $\Pi_P C(P)$ can be chosen as the M of the preceding discussion. Thus, M can be selected to be an invariant divisor of K/k. The intersection $R(M)$ of all groups E defined above for all P yields the generalization of the ray used in class field theory. Local class field theory yields

$$[FE:TE] = e(\mathfrak{p}),$$

where $e(\mathfrak{p})$ is the ramification degree of $\mathfrak{p}$ in a typical K_P.

Moreover, we require information on factor sets $U_{\sigma,\tau}$ of units U in K. Applying the methods used in section 3 we find

$$[FU^* : TU^*] = [\Gamma : \Gamma']^r 2^\rho$$

for the Herbrand subgroup U^* of U.[10] Here Γ' denotes the commutator group of Γ, r, the number of basal units $u = U \frown k$ which lie in the ground field k, ρ the number of real (infinite) prime divisors of k that are ramified in K. In order to evaluate $[FU : TU]$ we make repeated use of Herbrand's Lemma and find that the essential index of this reduction is

$$[FU : FU^*] = [U : U^*]^{n-1} [U \frown k : U^* \frown k] \cdot L$$

where L denotes a rather complicated factor of reduction. Examples show that L can be different from unity, although in the cyclic case the analogous index can be proved to be one, by the Herbrand Lemma on groups. Combining the various results we ultimately find

$$[F\mathfrak{A}_M : T\mathfrak{A}_M \cdot F_M(A_M)] = [FA_M : F(A_M)] [\Gamma : \Gamma'] h_K^{1-n} h_k^{-1} L^{-1},$$

where L and Γ' are defined as above, while

$$h_K = [\mathfrak{A} : (A)], \qquad h_k = [\mathfrak{A} \frown k : (A \frown k)]$$

are the respective class numbers of K and k. (Here A denotes the group of all non-zero numbers of k, (A), the group of all principal ideals of K.)

5. *Generalized Artin Symbol and Norm Residue Symbol.*—Finally, by means of our results, we can define a generalization of the classical symbols of class field theory. They yield formal implications of our general theory, which is based on the notions of ideal algebras and classes of algebras. These symbols can be used to describe the law of decomposition of unramified prime divisors.[11]

[1] C. Chevalley, "Généralisation de la théorie du corps de classes pour les extensions infinis," *Jour. Math. pures et appliques*, ser. 9, **15**, 359–371 (1936); A. Weil, "Remarques sur les résultats recents de C. Chevalley," *C. R.*, Paris, **203**, 1208 (1936); A. Weil, "Zur algebraische Theorie der algebraischen Funktionen," *Jour. reine ang. Math.*, **179**, 129–133 (1938).

[2] As usual, a prime divisor of k is simply a valuation of k. These valuations are of two types: "finite" and "infinite." Each prime ideal $\mathfrak{p}$ gives a finite prime divisor, for which the valuation function is $V_\mathfrak{p}(\alpha)$, the exact exponent to which α is divisible by $\mathfrak{p}$. The infinite prime divisors correspond to archimedean valuations of k given by the various subfields of the complex field conjugate to k.

[3] For general information, see: A. A. Albert, "Structure of Algebras," *Am. Math. Soc.* Col. Pub., **24** (1939).

[4] $\mathfrak{p}$ is a ramification divisor of H if $m_\mathfrak{p} \neq 1$, and conversely.

[5] M. Hall, "Group Rings and Extensions I," *Ann. Math.*, **39**, 220–234 (1938); T.

Nakayama, "Ueber die Beziehungen der Faktorensystemen und der Normenklassengruppe eines galoisschen Erweiterungskörpers," *Math. Ann.*, **112**, 85–91 (1935).

[6] If R is any multiplicative abelian group with the operators of Γ_I, then FR will denote the group of all factor sets $R_{\sigma,\tau}$ with components $R_{\sigma,\tau}$ in R and subscripts σ, τ ranging over Γ. In like manner TR designates the group of all transformation sets $R_\sigma R_\tau^\sigma / R_{\sigma,\tau}$ formed from vectors R_σ over the group R.

[7] E. Noether, "Der Hauptgeschlechtssatz für relativgaloissche Zahlkörper," *Math. Ann.*, **108**, 411–419 (1932).

[8] Here, as previously, a subscript "M" signifies that one is to consider only those elements prime to M in the group concerned.

[9] C. Chevalley, "Sur la theorie du corps de classes dans les corps finis et les corps locaux," *Jour. Fac. Sci., Univ. Tokyo*, **2**, 365–476 (1933); H. Hasse, Mimeographed lecture notes on Klassenkörpertheorie, Marburg, 1933.

[10] J. Herbrand, "Nouvelle démonstration et généralisation d'un théorème de Minkowski," C. R., Paris, **191**, 1282 (1930).

[11] R. Brauer, H. Hasse and E. Noether, "Beweis eines Hauptsatzes in der Theorie der Algebren," *Jour. reine angewandte Math.*, **167**, 399–404 (1932).

ON AN ERGODIC ANALYSIS OF THE REMAINDER TERM OF MEAN MOTIONS

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Let $a_1, \ldots, a_n$ be positive constants for which $b_1 + \ldots + b_k \neq b_{k+1} + \ldots + b_n$, $(k = 1, 2, \ldots, n-1)$ if $(b_1, \ldots, b_n)$ is any permutation of $(a_1, \ldots, a_n)$. Denote by $z(t)$ the trigonometric sum

$$z(t) = \sum_{j=1}^{n} a_j e(\lambda_j t + \phi_j^0), \qquad (e(u) = \exp 2\pi i u),$$

and suppose that the frequencies $\lambda_1, \ldots, \lambda_n$ are fixed and linearly independent, while the initial phases $\phi_1^0, \ldots, \phi_n^0$ are arbitrary on the n-dimensional torus T: $0 \leqq \phi_j^0 < 1$. One can exclude from T a $(n-1)$-dimensional ϕ^0-hypersurface in such a way that for the remaining points of T the function $z(t)$ does not vanish for $-\infty < t < +\infty$; so that the pair of conditions

$$e(\psi(t)) = z(t)/|z(t)|; \qquad 0 \leqq \psi(0) < 1,$$

determines a unique regular function ψ of t along the t-axis. By an application of Birkhoff's ergodic theorem,[1] it was recently shown[2] that $\psi(t)$ has, for almost all choices of the initial phases on T, a mean motion

$$\mu = \lim_{t \to \infty} \psi(t)/t, \text{ i.e., } \mu = M\{\psi'(t)\}, \text{ where } M\{g(t)\} = \lim_{T \to \infty} v^{-1} \int_0^v g(t)dt$$

(and $\psi'(t) = d\psi(t)/dt$); so that the arcus $\psi(t)$ may be analyzed into a secular term $\mu.t$ and a remainder term $\omega(t) = \psi(t) - \mu.t$ of lower order.

In view of the applications, one would expect that the remainder term $\omega(t)$ admits of an anharmonic analysis, as tacitly assumed by the astronomers. However, the existence of such an analysis never was mathematically established (except in Lagrange's case $a_1 > a_2 + a_3 + \ldots + a_n$, in which case $\omega(t)$ turned out to be uniformly almost periodic[3]). The object of this note is to fill in this gap to some extent, by proving for $\lambda \neq 0$ the existence of the Fourier averages $F(\lambda) = M\{e(-\lambda t)\omega(t)\}$, $-\infty < \lambda < +\infty$, of $\omega(t)$, if, as before, a set of zero measure on T is excluded. The proof consists of a refinement of the considerations applied loc. cit.[2] In fact, Birkhoff's ergodic theorem will be applied, not once but infinitely often. What is actually proved in this manner is the almost periodicity (B) of $\omega'(t)$. The existence of $F(\lambda)$ then follows from the existence of $G(\lambda) = M\{e(-\lambda t)\omega'(t)\}$ and from the relation

$$2\pi i F(\lambda) = G(\lambda)/\lambda, \qquad (-\infty < \lambda < +\infty, \lambda \neq 0;\ G(0) = 0),$$

which may be established by using the existence of μ.

It will also follow that $G(\lambda)$ is, for fixed λ, homogeneous and linear in the given frequency $\lambda_1, \ldots, \lambda_n$ of $z(t)$, and that $G(\lambda) = 0$ unless λ is of the form $m_1\lambda_1 + \ldots + m_n\lambda_n$, where $m_j = 0, \pm 1, \pm 2, \ldots$ for $j = 1, \ldots, n$. Hence, it is seen from the preceding connection between G and F, that the Fourier averages of $\omega(t)$ involve the small divisors of classical celestial mechanics,[4] or, equivalently, the Diophantine problematics of irrational rotation numbers in the general stability problem of Birkhoff.[5] Thus, the Fourier analysis of the remainder term $\omega(t)$ of $\psi(t) = \mu.t + \omega(t)$ automatically leads, at least formally, to the central unsolved problem of general dynamics. This might be connected with the circumstance that, although the derivative $\omega'(t) = \psi'(t) - \mu$ turns out to be almost periodic (B), the almost periodicity (B) of $\omega(t)$ itself will remain undecided.

First, if T denotes the n-dimensional torus $0 \leqq \phi_j < 1$, $(j = 1, \ldots, n)$, and if the amplitudes a_j and the frequencies λ_j of $z(t)$ are fixed, then, on placing

$$\chi(\phi_1, \ldots, \phi_n) = \text{real part of } \sum_{j=1}^{n} \lambda_j a_j e(\phi_j) / \sum_{j=1}^{n} a_j e(\phi_j),$$

one obtains a function χ of the position $(\phi_1, \ldots, \phi_n)$ on T. It was proved loc. cit.[2] that χ is integrable (L) over T. On expressing $\exp i\alpha$ in terms of $\cos\alpha$ and $\sin\alpha$, one readily sees that $\chi(-\phi_1, \ldots, -\phi_n) = \chi(\phi_1, \ldots, \phi_n)$. Hence, the Fourier series (L) of χ on T is of the form

$$\chi(\phi_1, \ldots, \phi_n) \sim \sum_m A(m_1, \ldots, m_n) \cos 2\pi(m_1\phi_1 + \ldots + m_n\phi_n),$$

where the n summation indices $m_1,\ldots,m_n$ run from $-\infty$ to $+\infty$. (Since χ is homogeneous and linear in $(\lambda_1,\ldots,\lambda_n)$, the same holds for each of the Fourier constants A.) Let k denote the collection $(k_1,\ldots,k_n)$ of n positive integers k_j, and $\sigma_k = \sigma_k(\phi_1,\ldots,\phi_n)$ the n-fold trigonometric polynomial which is the kth partial sum of the Fourier series of χ. Then, since χ is integrable (L), one has

$$\int_T |\chi(\phi_1,\ldots,\phi_n) - \sigma_k(\phi_1,\ldots,\phi_n)|dT \to 0 \text{ as } k \to \infty,$$

if $k \to \infty$ means that $k_j \to \infty$; $j = 1,\ldots,n$.

Consider the cyclic transformation group

$$\phi_1 = \lambda_1 t + \phi_1^0,\ldots,\phi_n = \lambda_n t + \phi_n^0; \quad -\infty < t < +\infty.$$

This group of transformations of T into itself is metrically transitive in view of the linear independence of $\lambda_1,\ldots,\lambda_n$ (Kronecker-Weyl). Furthermore, the function $|\chi - \sigma_k|$ on T is integrable (L) for every fixed k. But $\sigma_k(\phi_1,\ldots,\phi_n)$ becomes, in virtue of the cyclic group, an almost periodic polynomial $s_k(t) = \sigma_k(\lambda_1 t + \phi_1^0,\ldots,\lambda_n t + \phi_n^0)$, while the definition of the function χ on T implies that $\chi(\lambda_1 t + \phi_1^0,\ldots,\lambda_n t + \phi_n^0)$ is identical with the derivative $\psi'(t)$ of the angular function $\psi(t)$, defined in the introduction. It follows therefore from the ergodic theorem of Birkhoff that, if k is arbitrarily fixed and if the initial phase $(\phi_1^0,\ldots,\phi_n^0)$ on T is chosen arbitrarily but not on a certain zero set (which, if it exists at all, might depend on k), then the time average $M\{|\psi'(t) - s_k(t)|\}$ exists and is identical with the space average of $|\chi - \sigma_k|$ over T. But, as was pointed out before, this space average tends to zero as $k \to \infty$; so that, since for each of the integral indices $k = (k_1,\ldots,k_n)$ only a zero set of initial phases is excluded, the relation

$$M\{|\psi'(t) - s_k(t)|\} \to 0, \qquad k \to \infty,$$

holds whenever the initial phase $(\phi_1^0,\ldots,\phi_n^0)$ is not in a zero set of T.

Since $s_k(t)$ is an almost periodic polynomial, it follows that $\psi'(t)$ is almost periodic (B) for almost all choices of the initial phase. Furthermore, it is clear from the definition of $s_k(t)$ that the anharmonic Fourier analysis (B) of $\psi'(t)$ may be obtained by writing $\lambda_1 t + \phi_1^0, \ldots,\lambda_n t + \phi_n^0$ for $\phi_1,\ldots,\phi_n$ in the Fourier series (L) of the periodic function $\chi(\phi_1,\ldots,\phi_n)$.

Finally, the passage from $\omega'(t) = \psi'(t) - \mu$ to $\omega(t) = \psi(t) - \mu t$, as described in the introduction, requires merely a cautious application of partial integrations on the average.

It was suspected loc. cit.[2] (p. 263) that, as far as the existence of the constant μ is concerned, the exceptional zero set of the ergodic theorem is vacuous, and Weyl[6] has succeeded in proving, among other things, that such is actually the case. However, it remains undecided whether or not

the same is true in case of the finer problem of the present note. On the other hand, there is no difficulty at all in transferring the above considerations to the case where the frequency $\lambda_1, \ldots, \lambda_n$ of $z(t)$ are not linearly independent.[7]

[1] G. D. Birkhoff, *Proc. Nat. Acad. Sci.*, **17**, 656–660 (1931).

[2] P. Hartman, E. R. van Kampen, and A. Wintner, *Amer. Jour. Math.*, **59**, 261–269 (1937).

[3] A. Wintner, *Rend. R. Accad. Naz. Lincei*, (6) **11**, 464–467 (1930).

[4] Cf. A. Wintner, *Math. Zeit.*, **31**, 434–440 (1930) and *Amer. Jour. Math.*, **59**, 801 (1937).

[5] Cf. G. D. Birkhoff, *Ann. Inst. Poincaré*, **2**, 369–386 (1932).

[6] H. Weyl, *Amer. Jour. Math.*, **60**, 889–896 (1938).

[7] Cf. loc. cit., reference 2, 263; H. Weyl, *Ibid.*, **61**, 143–148 (1939)

GROUPS WHICH CONTAIN LESS THAN FOURTEEN PROPER SUBGROUPS

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The earliest fundamental theorem relating to the development of the groups of finite order was announced in an incomplete form by J. L. Lagrange (1736–1813) and asserts that the order of a group is divisible by the order of each of its subgroups. The work of J. L. Lagrange related directly only to the subgroups of the symmetric group but the method of the proof in this special case is easily extended to the general case and hence the general theorem has sometimes been called Lagrange's theorem relating to group theory. It may be noted that this was just about one hundred years earlier than the announcement of Sylow's theorem in 1872, which incorrectly is said to be the first to be proved among a set of closely related theorems on the structure of finite groups, in the *Encyclopaedia Britannica*, volume 10, page 914 (1938). The present article is one of a series of recent contributions along primitive lines on the subject of finite groups.

The groups which contain less than twelve proper subgroups were determined in these PROCEEDINGS, **25**, 540–543 (1939). If an abelian group contains exactly twelve proper subgroups its order cannot be divisible by as many as three distinct prime numbers since such a group is the direct product of its Sylow subgroups. If its order is divisible by two distinct prime numbers it is the cyclic group of order $p_1 p_2^6$, p_1 and p_2 being distinct prime numbers. If it is a non-cyclic prime power group one of its invariants is a prime number p, and therefore the number of its proper subgroups

is $(k-1)(p+1)$, p^k being the order of the group. Hence there is one and only one such group of each of the following orders: 32, 81, 125, 121. These correspond to the following pairs of values of p and k, respectively: 2, 5; 3, 4; 5, 3; 11, 2. When the group is cyclic its order is p^{13}. Hence *there are exactly six abelian groups which separately involve twelve proper subgroups*. Two of these are cyclic and consist of infinite systems of groups.

If G is a non-abelian prime power group which contains exactly twelve proper subgroups its order cannot be 16 and if its order is 32 it is the group of this order which contains exactly two non-invariant subgroups. Hence there is one and only one non-abelian group which contains exactly twelve proper subgroups and has an order which is of the form 2^m. If a non-abelian group whose order is of the form 3^m contains exactly twelve proper subgroups it contains an invariant subgroup of order 27 which involves operators of order 9. This subgroup involves eight proper subgroups. Hence it must contain operators of order 27 and be the group of order 81 which involves three and only three non-invariant subgroups. From similar considerations it results that if a non-abelian group whose order is of the form 5^m contains exactly twelve proper subgroups it is the group of order 125 which contains exactly five non-invariant subgroups of order 5. Since a non-abelian group whose order is a prime power of a larger prime number than 5 cannot contain exactly twelve proper subgroups it has been proved that *there are exactly three non-abelian prime power groups which separately contain exactly twelve proper subgroups*. The orders of these three groups are 32, 81 and 125, respectively, and they are all conformal with abelian groups.

When the order of a non-abelian group G which contains exactly twelve proper subgroups is divisible by two distinct prime numbers the larger of these prime numbers is less than 13, for G is not the direct product of its Sylow subgroups. If the larger of these prime numbers is 11 then G contains only one subgroup of order 11 and its order is either 22 or 55, being either the dihedral group of order 22 or the semi-metacyclic group of order 55. If the larger of the two prime factors of the order of G is 7 then G contains only one subgroup of order 7 and it is formed by establishing a 7, 4 isomorphism between the dihedral group of order 14 and the cyclic group of order 8 or by establishing a 7, 9 isomorphism between the semi-metacyclic group of order 21 and the cyclic group of order 27. If the larger of the two prime factors of the order of G is 5 then G contains an invariant subgroup of order 5 and is formed by a 5, 8 isomorphism between the dihedral group of order 10 and the cyclic group of order 16, or it is the metacyclic group of order 20.

If an abelian group contains exactly thirteen proper subgroups its order is divisible by at most two distinct prime numbers and the cyclic group of order $p_1^2 p_2^4$, p_1 and p_2 being distinct prime numbers, is the only group whose

order is divisible by two distinct prime numbers which has exactly thirteen proper subgroups. The direct product of the four group and the cyclic group of order p^2, p being an arbitrary odd prime number, is the only non-cyclic abelian group which contains exactly thirteen proper subgroups and has an order which is divisible by two distinct prime numbers. If a prime power abelian group contains exactly thirteen proper subgroups it is the group of order 16 and of type 2, 2 if it has two invariants and the cyclic group of order p^{14} if it has only one invariant. Hence there are four and only four abelian groups which separately involve thirteen and only thirteen proper subgroups. Three of these are infinite systems of groups while one is an individual group.

There are two non-abelian groups of order 16 which separately contain exactly thirteen proper subgroups. One of these is obtained by extending the cyclic group of order 8 by operators which transform each of its operators into the third power and the other is obtained by extending the abelian group of type 2, 1 by an operator of order 4 which transforms each of its operators into its inverse but has a different square than its operators of order 4. If a non-abelian group of order 32 would contain exactly thirteen proper subgroups it could not involve a cyclic group of order 16 and therefore each of its subgroups of order 16 would contain at least nine proper subgroups. Hence there are two and only two non-abelian groups which separately contain thirteen proper subgroups and have an order which is a power of 2. It is easy to see that a non-abelian group whose order is a power of an odd prime number cannot contain exactly thirteen proper subgroups and hence there are two and only two prime power non-abelian groups which separately involve exactly thirteen proper subgroups.

If the order of a non-abelian group which contains exactly thirteen proper subgroups is divisible by just two distinct prime numbers these prime numbers are 2 and 3, respectively. There are four Sylow subgroups whose order is a power of 3 in such a group, because if G contained only one such Sylow subgroup this subgroup could not be cyclic and of order as large as 9 because the dihedral group of order 18 contains fourteen proper subgroups. It could not be of order 3 since the corresponding quotient group would have for its order a power of 2 and could not be cyclic because G contains an odd number of proper subgroups. It could not be non-cyclic because there would be a subgroup of index 2 under G which would be the direct product of the group of order 3 and an abelian group whose order is a power of 2. As this is impossible G contains four subgroups whose order is a power of 3 and transforms them according to the tetrahedral group.

If these four subgroups are of order 9 they are cyclic and G is formed by a 4, 3 isomorphism between the tetrahedral group and the cyclic group of order 9. If the given four Sylow subgroups are of order 3 they generate the non-twelve group of order 24 which is known to contain thirteen proper

subgroups. The order of G cannot be divisible by as many as three distinct prime numbers since G could not be the direct product of two groups and a non-abelian group cannot contain exactly five proper subgroups. A simple group of composite order could not contain exactly thirteen proper subgroups since its order could not be divisible by more than two distinct prime numbers for there would be more than one Sylow subgroup of each order. The smallest possible order of G would be the product of powers of 2, 3, 5 and hence there would be at least $3 + 4 + 6$ Sylow subgroups, but there would also be subgroups which would not be Sylow subgroups since if there were only 3 subgroups whose orders would be powers of 2 they would be transformed according to a group of degree 3 and hence G could not be simple. If the order of G would be divisible by as many as three distinct prime numbers the largest of these numbers would be at least equal to 5. This is impossible, and hence it results that *there are exactly eight different groups which separately contain thirteen proper subgroups.* The four non-abelian groups which have this property consist of two groups of order 16, one of each of the orders 24 and 36.

ON THE DECOMPOSITION OF TRANSITIVE PERMUTATION GROUPS GENERATED BY THE SYMMETRIC GROUP

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1. *Introduction.*—The double cosets $H_\alpha S H_\alpha$ into which the symmetric group G of degree n is decomposed with respect to a subgroup H_α are associated in an interesting way with the irreducible representations $\{\gamma\}$ which occur as components in the reduction of the permutation group $G_H(\alpha)$. The subgroup $H = H_\alpha$ is taken to be the direct product of symmetric groups of orders $\alpha_1, \alpha_2, \ldots \alpha_n$; and $G_H(\alpha)$ is the transitive permutation group on the cosets $H_\alpha S_i$ which orders to an element R of G the permutation $H_\alpha S_i \rightarrow H_\alpha S_i R = H_\alpha S_j$. If we restrict R to the elements of H_α, the cosets which are then permuted among themselves form the aggregates $H_\alpha S_i H_\alpha$ called double cosets. Now if $G_H(\alpha)$ is written as a group of permutation matrices and completely reduced, and if μ_γ^α denotes the multiplicity of the irreducible representation $\{\gamma\}$ in this reduction, then we have

$$G_H(\alpha) = \sum_\gamma \mu_\gamma^\alpha \cdot \{\gamma\}. \qquad (1)$$

The number of Hermitian invariants of this group is equal on the one hand to the number of double cosets $H_\alpha S H_\alpha$ and on the other hand to the sum

of the squares of the multiplicities μ_γ^α of the irreducible components $\{\gamma\}$, so we have

$$\sum_\gamma (\mu_\gamma^\alpha)^2 = \text{number of double cosets } H_\alpha S H_\alpha. \qquad (2)$$

We propose to prove the following theorems.

THEOREM I.—The double cosets $H_\alpha S H_\alpha$ of the symmetric group G (with respect to a subgroup H_α which is the direct product of symmetric groups) can be displayed in a series of squares, one for each irreducible component $\{\gamma\}$ of $G_H(\alpha)$, so that each square shall have its $(\mu_\gamma^\alpha)^2$ double cosets so arranged that inverse double cosets are symmetrically placed with respect to the diagonal, and self-inverse double cosets occupy the diagonal positions.

THEOREM II.—The number of self-inverse double cosets $H_\alpha S H_\alpha$ is equal to the number of irreducible components of $G_H(\alpha)$, each counted according to its multiplicity.

2. *Partitions and Tableaux.*—Each partition $[\alpha]$ of an integer n into a sum of positive integers,

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n = n,\ \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \ldots \geq \alpha_n > 0, \quad (3)$$

may be associated with an $[\alpha]$-tableau having α_i dots (or symbols, or digits) in the ith row, $i = 1, 2, \ldots r$. Frobenius and others have associated such tableaux with sets of conjugate elements and with irreducible representations of the symmetric group G of degree n. We may further associate this tableau both with the subgroup H_α of G which permutes symmetrically among themselves the symbols in each of the rows of the tableau, and also with the transitive permutation group $G_H(\alpha)$ which gives a reducible linear representation of the symmetric group.

Following Littlewood, Richardson[1] and others, G. deB. Robinson[2] has used these tableaux in a study of the decomposition of the direct product of irreducible representations of the symmetric group. His representation of the elements of the symmetric group in a series of squares corresponding to the irreducible representations suggested the theorems of this paper about the double cosets $H_\alpha S H_\alpha$. For when the subgroup H_α is the identity subgroup, the double cosets become the elements themselves, and his diagram becomes a special case of ours.

3. *Standard Tableaux and Double Cosets.*—We shall use the expression "normal $[\alpha]$-sequence" to denote a sequence of α_1 1's, α_2 2's, α_3 3's, ... $\alpha_r r$'s, in that order, and the expression "normal $[\alpha]$-tableau" to denote the same sequence arranged in r rows to form an $[\alpha]$-tableau. The substitution $x_i \rightarrow x_j$ operating on a sequence or tableau is to be thought of as carrying the symbol in the ith position into the jth position. The subgroup H_α of G obviously leaves a normal $[\alpha]$-sequence (or tableau) invariant, and each

element of a coset $H_\alpha S$ transforms it into the same deranged $[\alpha]$-sequence (or tableau). The substitutions from the double coset $H_\alpha S H_\alpha$ shuffle among themselves the digits or symbols of a given row of the deranged tableau corresponding to $H_\alpha S$, but they all leave the same number $c_j^i(S)$ of digits j in the ith row. A particular substitution of $H_\alpha S H_\alpha$ can be found which will arrange the digits in non-increasing order within each row, and we shall suppose that S itself is so chosen. The resulting tableau will be called a standard $[\alpha]$-tableau, and the corresponding sequence obtained by writing the successive rows of this tableau in a single line, separated by commas, will be called a standard $[\alpha]$-sequence. Both will be denoted by $[\alpha]^S$. There is thus a one-to-one correspondence between standard $[\alpha]$-sequences $[\alpha]^S$ and double cosets $H_\alpha S H_\alpha$. Both may be defined briefly by the $r \times r$ matrix $c_j^i(S)$, for which we have

$$\sum_{j=1}^{r} c_j^i(S) = \sum_{j=1}^{r} c_i^j(S) = \alpha_i, \; i = 1, 2, \ldots r. \tag{4}$$

If a substitution S takes a digit j from a normal $[\alpha]$-tableau into the ith row, the inverse substitution S^{-1} will take a digit i into the jth row, so we have

$$c_j^i(S) = c_i^j(S^{-1}). \tag{5}$$

It will be convenient to use the symbol $[\alpha]_\gamma$ to denote a tableau (or sequence) whose digits are taken from a normal $[\gamma]$-tableau, but are arranged in the form of a standard $[\alpha]$-tableau (or corresponding sequence), so that the digits are non-increasing in each of the rows of the latter. Thus, for example, let $n = 7$, $[\alpha] = [3,2,1,1]$, $[\gamma] = [5,1,1]$ and let S_1 be the permutation $(a)(bdecf)(g)$. We chose $S = H_\alpha S_1 H_\alpha = (ac) \cdot S_1 \cdot (acb)(de)(f)(g) = (af)(be)(c)(d)(g)$ so that $[\alpha]^S$ is standard, and obtain the following tableaux:

NORMAL $[\alpha]$;	DERANGED $S_1[\alpha]$;	STANDARD $[\alpha]^S$;	$[\alpha]_\gamma^S$;
111	132	321	111
22	12	21	21
3	1	1	1
4	4	4	3

(6a)

and the corresponding sequences:

111, 22, 3, 4; 132, 12, 1, 4; 321, 21, 1, 4; 111, 21, 1, 3. (6b)

The method for transforming the $[\alpha]^S$-sequence to the $[\alpha]_\gamma^S$-sequence will be described in the next paragraph.

4. *Lattice Permutations and Sequences.*—A deranged sequence is said to be a lattice permutation of a normal sequence if among the first k digits, $k = 1, 2, \ldots, n$, the frequency of any digit does not exceed the frequency of any lower digit. The procedure described by Robinson[2] and his prede-

cessors may be used to transform in a unique manner a standard $[\alpha]$-sequence into a lattice $[\alpha]_\gamma$ sequence, i.e., into a sequence which is a lattice permutation of a normal $[\gamma]$-sequence but has the form of a standard $[\alpha]$-sequence. We may summarize their method as follows.

If we denote by $m_{k,s}$ the number of times that the digit s occurs among the first k digits of the sequence, then if the sequence is non-lattice, some of the differences $d_{k,s} = m_{k,s} - m_{k,s-1}$ must be positive. Pick the smallest $s \geq 2$ for which some $d_{k,s} > 0$, then the largest difference $d_{k,s}$ for that s, and then the smallest k for that difference. Change this kth digit from s to $s - 1$. If this change makes some $d_{l,s-1} = 1$, reapply the same rules to change the lth digit from $s - 1$ to $s - 2$, etc., and suppose that finally when we change a $t + 1$ to a t in this way that the digits less than s are again in lattice order. We now have a new tableau with $\alpha_s - 1$ digits s and $\alpha_t + 1$ digits t, and we associate with this reduction the operator C_{ts}. Starting all over again on the new tableau or sequence, if it is not already lattice, we make successive reductions until finally the standard $[\alpha]$-sequence is reduced to a lattice $[\alpha]_\gamma$-sequence. With this entire transformation we associate the operator

$$L_\lambda = \prod_{j=2}^{r} \prod_{i<j} C_{ij}^{\lambda_{ij}}, \tag{7}$$

where λ_{ts} denotes the number of digits s which are replaced by t according to the procedure just described. For $t > s$ it is convenient to define $\lambda_{ts} = -\lambda_{st}$, so as to make the matrix (λ_{ij}) skew-symmetric. The new partition $[\gamma]$ is then given in terms of $[\alpha]$ and λ by the equations

$$\gamma_i = \alpha_i + \sum_{j=1}^{r} \lambda_{ij}. \tag{8}$$

It may of course happen that the last r-r' digits γ_i may vanish. The inverse of the operator L_λ is defined by the transpose of the skew-symmetric matrix (λ_{ij}).

The number of solutions of (8) for given partitions $[\alpha]$ and $[\gamma]$ is known[3] to equal μ_γ^α, the number of times the irreducible representation $\{\gamma\}$ corresponding to the partition $[\gamma]$ occurs in the reduction of the permutation group $G_H(\alpha)$.

We are concerned in this proof with the reduction to lattice form of the standard $[\alpha]$-tableaux corresponding to an arbitrary but definitely chosen standard permutation S and its inverse S^{-1}. We shall denote these tableaux and the corresponding sequences by A and A'. We shall further distinguish between various operators C_{ts} by letting $C_{ts,r} = C_t(s,s',s'', \ldots s^{(p)}; r,r',r'', \ldots r^{(p)})$ be an operator which starts by reducing a digit s in row r of the tableau, is thereby forced to reduce a digit in row $r' \geq r$ whose original value was $s' \leq s$, then a digit in row $r'' \geq r'$ whose original value

was $s'' \leq s'$, etc., and finally reduces to the value t a digit in row $r^{(p)} \geq \ldots r'' \geq r' \geq r$, whose original value was $s^{(p)} \geq \ldots s'' \geq s' \geq s$.

The following example will illustrate the reduction just described. Let $n = 11$, and $[\alpha] = [3,3,2,2,1]$. The permutations

$$S\colon\ (afeigkdj)(bc)(h) \quad \text{and} \quad S^{-1}\colon\ (ajdkgief)(bc)(h) \tag{9}$$

generate double cosets whose standard $[\alpha]$-tableaux A, A', and matrices $c_j^i(S) = c_i^j(S^{-1})$ are the following:

$$A\colon \begin{array}{l} 411 \\ 521 \\ 43 \\ 22 \\ 3 \end{array} \qquad A'\colon \begin{array}{l} 211 \\ 442 \\ 53 \\ 31 \\ 2 \end{array} \qquad c_j^i(S) = \begin{pmatrix} 2&0&0&1&0 \\ 1&1&0&0&1 \\ 0&0&1&1&0 \\ 0&2&0&0&0 \\ 0&0&1&0&0 \end{pmatrix}. \tag{10}$$

The reduction of the corresponding sequences is as follows.

A:	411, 521, 43, 22, 3 (standard)	A':	211, 442, 53, 31, 2	
	−11, −21, −−, 22, −	$C_{12,1}$	**1**11, −−2, −−, −1, 2	
	−11, −21, −3, 22, 3	$C_{23,4}$	111, −−2, −3, **2**1, 2	(11)
$C_{14,3}$	−11, −21, **32**, **2**1, 3	$C_{24,2}$	111, −**2**2, −3, 21, 2	
$C_{14,1}$	**1**11, −21, 32, 21, 3	$C_{14,2}$	111, **2**22, −3, **1**1, 2	
A_γ: $C_{25,2}$	111, **2**21, 32, 21, 3 (lattice)	A'_γ: $C_{35,3}$	111, 222, **3**3, 11, 2	
	111, 222, 33, 44, 5 (normal)		111, 222, 33, 44, 5	

$$L_\lambda = C_{14,3}\, C_{14,1}\, C_{25,2}. \qquad L_{\lambda'} = C_{12,1}C_{23,4}C_{24,2}C_{14,2}C_{35,3}.$$

At each stage the digits $\geq s$ are indicated by dashes, and the newly changed digits $< s$ are shown in bold face type, with the corresponding operator $C_{ts,\,r}$ at the left. The final lattice sequences A_γ and A'_γ are followed by the normal $[\alpha]$-sequence for purposes of comparison. It is readily seen in this example (11) that the change from the normal $[\alpha]$-sequence to the lattice A_γ sequence is essentially given by the same operator $L_{\lambda'}$ which takes A' into A'_γ, and that the change from the normal $[\alpha]$-sequence to A'_γ is given by L_λ, if a factor $C_{ts,\,r}$ is now interpreted to mean that a digit t appears in row s. The new significance for the subscript r is that when $r \leq s$ there is a digit r in A which reduces to the digit t in row s of A, and corresponds to the operator $C_{ts,\,r}$ of $L_{\lambda'}$. If $r > s$, however, and if there are f_s digits s in the tableau A_γ, which are in row s, then a final digit $t < s$ in A occurs f_s places to the right of the corresponding r in A, in order that the standard form be preserved.

That this dual relationship of the operators L_λ and $L_{\lambda'}$ is true in general will be shown in §5. In the special case considered by Robinson,[2] this was considered obvious. But in the more complicated situation treated in this paper, some indication of the proof seems to be required. From this

duality we see that the operators L_λ and $L_{\lambda'}$ together determine the standard tableaux corresponding to both S and S^{-1}, their rôles being merely interchanged in the two cases. This is the crucial fact which will be used in §6 to establish our theorems.

5. *The Duality of the Operators for S and S^{-1}.*—We concentrate our attention on the effect of a factor $C_{ts,r}$ of $L_{\lambda'}$ in the reduction of A'. At each stage in the reduction process, the need for reducing a digit s in row r is determined completely by the original arrangement of digits $\leq s$ in rows $<r$, and other digits s which precede it in row r. Also a digit s in row r of A' corresponds to a certain digit r in row s of A, and we shall see that if the operator $C_{ts,r}$ is applied to this digit s in the reduction of A', then the corresponding digit r in A will give rise to a digit t in the same row s of A. Let us first analyze the cases $s = 2,3$. There are certain critical digits 2 in A' which reduce to 1. An operator $C_{12,r}$ changing a critical digit 2 to 1 in row $r > 2$ of A' will correspond to a change from r to 2 and 2 to 1 in row 2 of A, since at this stage in the reduction of A all digits $< r$ in row 1 have become 1's, all the critical digits $< r$ in row 2 have become 1's (as may be assumed by induction), and all other digits $<r$ in row 2 have become 2's. Hence at the stage when this digit r is reduced, the 2's in row 2 just balance the 1's in row 1, and there are no digits between 2 and $r - 1$ in row 1. Consequently this r must reduce to 2, and then the last 2 in the second row must reduce to 1.

The case $s = 3$ is somewhat more typical. An operator $C_{23,r}$ changing a 3 to 2 in row $r \geq 2$ of A' will correspond in A to a change from r to 2 or from r to 3 and 3 to 2 in row 3. At this stage there are as many digits $\leq r$ which have become 3's in row 3 as digits $<r$ which have become 2's in row 2, but less digits $< r$ which have become 2's in row 2 than digits $< r$ which have become 1's in the first two rows. Furthermore, as the digits $\geq r$ in the first three rows are successively reduced, at no stage will an excess of 2's be created in the third row which will force this 2 to become a 1. Hence each operator $C_{23,r}$ for A' yields a 2 in row 3 of A. The operator $C_{13,r}$ changing 3 to 1 will do so directly in A' if $r = 1$, but in rows $r \geq 2$, there may be a change from 3 to 2 in row r which causes a change from 2 to 1 in a different digit of some row $r' \geq r$. Then the original value of this new digit must have been $s' \leq 3$. To this situation in A' will correspond in A a digit r in row 3 which is reduced to 2, and a digit $r' \geq r$ in row $s' \leq 3$ which is subsequently reduced to 2, thereby causing an excess of 2's which forces a 2 in row 3 down to a 1. Hence each operator $C_{13,r}$ belonging to A' yields a 1 in row 3 of A.

In general let the operator $C_{ts,r} = C_t(s, s', s'', \ldots s^{(p)};\ r, r', r'' \ldots r^{(p)})$ be applied at the proper stage to the tableau A'. Then the reduction of the digit s in row r of A' may cause other digits further to the right in our sequence to be reduced in succession, until finally the last of them is reduced

to the value t. Let the original digits occupying these critical positions in A' be denoted by $s', s'', \ldots s^{(p)}$, and let their rows be $r', r'', \ldots r^{(p)}$. Then as stated in §4 the values of $r^{(k)}$ are non-decreasing, while those of $s^{(k)}$ are non-increasing. The first statement is obvious from the rules of lattice reduction; the second statement, though less obvious can be verified by studying the preceding stages at which the digits $s^{(k)}$ were reduced. If $s > r$, then our original digit s must be reduced directly at least as far as r, since all digits between r and $s - 1$ inclusive have already been eliminated from the first $r - 1$ rows. Now our critical digits $s^{(k)}$ in row $r^{(k)}$ of A' correspond to certain digits $r^{(k)}$ in row $s^{(k)}$ of A. We select a subsequence (r_k, s_k) of the pairs $r^{(k)}, s^{(k)}$ so as to eliminate equalities among the digits $s^{(k)}$, and also to eliminate equalities among the $r^{(k)}$ when $r^{(k)} < s^{(k)}$. If $s > r$, we reject s itself when $r = t$ or $r < r'$, but set $s = s_1$ when $r = r' > t$, and choose s_2 so that $r < r_2$. If $s \leq r$, we set $s = s_1$. If several of the $s^{(k)}$ are equal, we reject all but the one corresponding to the largest $r^{(k)}$. If $s^{(k)} > r^{(k)}$ and several $r^{(k)}$ are equal, we reject all but the largest of these $s^{(k)}$. We denote the resulting monotone decreasing subsequence of the $s^{(k)}$ by $s_1 > s_2 > \ldots s_q > t$, and the corresponding values of the $r^{(k)}$ by $r_1 \leq r_2 \leq \ldots \leq r_q$. These digits mark certain steps in the reduction of A' which correspond to the successive steps in the reduction of the digit r of row s in A. We find that q is the smaller of the two numbers $s - t$ and $r - t$. If $s > r$, then the $r - t$ reductions in the digit r of row s in A are caused successively by the reductions in the digits $r_1 \ldots r_q$ in the previous rows $s_1 \ldots s_q$. If $r > s$, then the digit r of row s in A is reduced directly to s, and the $s - t$ further reductions which take place either directly in this digit or in some other digit further to the left in the same row are caused by reductions in the digits $r_1 \ldots r_q$ which successively upset the lattice arrangement in digits less than s.

Thus we see that the operator $C_{ts,r}$ in the reduction of A' has as its counterpart in the reduction of A a succession of operations which change an r in row s of A to a t in row s of A_γ. Hence the tableau A which corresponds to the double coset generated by S is defined by the operators $L_{\lambda'}$ and L_λ, and in like manner the tableau A' corresponding to the inverse double coset is defined by these same operators taken in reverse order.

6. *Conclusion.*—In view of this duality between the operators L_λ and $L_{\lambda'}$, we may arrange all the double cosets $H_\alpha S H_\alpha$ whose standard $[\alpha]$-tableaux reduce to $[\gamma]$-tableaux belonging to the same partition $[\gamma]$ in a square array whose rows and columns may be designated by λ and λ', respectively. Then since we have $\lambda = \lambda'$ in the main diagonal, the diagonal double cosets will be self inverse, whereas in general the pairs of inverse double cosets will be symmetrically placed with respect to the diagonal. For each partition $[\gamma]$ we have as many rows in the square matrix of double cosets as there are operators L_λ, namely μ_γ^α. Consequently there are

$(\mu_\gamma^\alpha)^2$ double cosets associated with the partition $[\gamma]$ of n. Since the total number of double cosets $H_\alpha S H_\alpha$ is $\sum_\gamma (\mu_\gamma^\alpha)^2$, we have just one square of double cosets for each partition $[\gamma]$ which corresponds to an irreducible component $\{\gamma\}$ in the reduction of the permutation group $G_H(\alpha)$. If in this array we examine the placement of inverse double cosets, we see that Theorem I follows at once. If we count the number of self-inverse double cosets, we see that their number is $\sum_\gamma \mu_\gamma^\alpha$, and thus we prove Theorem II.

[1] Littlewood, D. E., and Richardson, A. R., *Phil. Trans. Roy. Soc. London* (A) **233**, 99–141 (1934).

[2] Robinson, G. deB., "On the Representations of the Symmetric Group," *Amer. Jour. Math.*, **60**, 745–759 (1938).

[3] Murnaghan, F. D., "The Theory of Group Representations," Johns Hopkins Press, Baltimore (1938), esp. pp. 150 ff.

GENERAL EXPANSION THEOREMS

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A set G of elements of a normed linear space[1] E is said to be fundamental if every element of E is the limit of a sequence of finite linear combinations of elements of G; it is called a base if every element x of E can be expressed uniquely in the form

$$x = \sum_{n=0}^{\infty} a_n y_n, \quad y_n \in G. \tag{1}$$

Theorems of G. D. Birkhoff,[2, 3] J. L. Walsh[4] and R. E. A. C. Paley and N. Wiener[5] illustrate the general principle that if the elements of two sets G_1 and G_2 are sufficiently near each other, in some suitable sense, then either set is fundamental, or is a base, if the other is. More precisely, the following theorem can be proved.

THEOREM 1. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences of elements of the normed linear space E. If there is a number λ, $0 < \lambda < 1$, such that for every sequence $\{a_n\}$ of complex numbers*

$$\left\| \sum_{n=1}^{N} a_n (x_n - y_n) \right\| \leqq \lambda \left\| \sum_{n=1}^{N} a_n x_n \right\|, \quad N = 1, 2, \ldots, \tag{2}$$

then if $\{x_n\}$ is a fundamental set or a base, $\{y_n\}$ has the same property

Theorem 1 for bases can be proved by a generalization of the proof given by Paley and Wiener when E is Hilbert space; a proof of Theorem 1 for fundamental sets is much simpler, and this weaker result suffices for many applications.

To show how Theorem 1 can be applied, I shall outline proofs of some theorems on the expansion of analytic functions of a complex variable in terms of functions $z^n[1 + h_n(z)]$, where the functions $h_n(z)$ are "small." These theorems include and generalize a number of expansions which occur in the literature; in some cases, the methods of this note lead to larger regions of validity for the expansions than have been obtained previously. One special case can be applied to a conjecture concerning the zeros of the derivatives of an entire function of exponential type.

For the space E, I take the set of functions $f(z)$, analytic in $|z| < r$ and continuous in $|z| \leqq r$, with the norm

$$||f|| = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right\}^{1/p}, \quad p \geqq 1 \tag{3}$$

G_1 is the set $\{z^n\}$, $n = 0, 1, 2, \ldots$; G_2 is a sequence of elements $\{g_n(z)\}$ of E. Then Theorem 1 yields

THEOREM 2. *If, for every sequence $\{a_n\}$ of complex numbers, some number p, $p \geqq 1$, and every N,*

$$\int_0^{2\pi} \Big|\sum_{n=0}^{N} a_n[z^n - g_n(z)]\Big|^p d\theta \leqq \lambda^p \int_0^{2\pi} \Big|\sum_{n=0}^{N} a_n z^n\Big|^p d\theta, \tag{4}$$

where $z = re^{i\theta}$, $\lambda < 1$, then any function $f(z)$, analytic in $|z| < r$ and continuous in $|z| \leqq r$, can be expanded in a series of the functions $g_n(z)$, the series converging uniformly in any circle $|z| \leqq r' < r$.

The direct deduction from Theorem 1 is that there exist coefficients c_n such that

$$\lim_{n\to\infty} \int_0^{2\pi} |f(re^{i\theta}) - \sum_{k=0}^{n} c_k g_k(re^{i\theta})|^p d\theta = 0. \tag{5}$$

For $z = r'e^{i\theta}$, $r' < r$, one can represent

$$f(r'e^{i\theta}) - \sum_{k=0}^{n} c_k g_k(r'e^{i\theta})$$

by Cauchy's integral along $|z| = r$, and apply Hölder's inequality to deduce the conclusion of Theorem 2 from (5).

Birkhoff's expansion theorem[3] can be deduced in a few lines from Theorem 2, with $1 < p \leqq 2$. Another theorem which can be obtained from

Theorem 2 resembles a theorem of S. Takenaka,[6] which has been shown[7] to contain a large number of special expansion theorems.

THEOREM 3. *If the functions $h_n(z)$ are analytic in $|z| < r_0$, vanish at $z = 0$, and have a common majorant $h(z)$ satisfying*

$$\frac{1}{2\pi}\int_0^{2\pi} |h(\rho e^{i\theta})|^2 d\theta \leqq K_\rho^2, \quad 0 < \rho < r_0; \tag{6}$$

then any function $f(z)$, analytic in $|z| < s$, where $s \leqq r_0$, can be expanded in a (uniformly convergent) series of the functions $z^n[1 + h_n(z)]$ in any circle

$$|z| \leqq s_1 < \min\left\{s, \sup_{0<\rho<r_0} \rho(K_\rho^2 + 1)^{-1/2}\right\}.$$

With $p = 2$, and $g_n(z) = 1 + h_n(z)$, (4) becomes

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{n=0}^{N} a_n r^n e^{in\theta} h_n(re^{i\theta})\right|^2 d\theta \leqq \lambda^2 \sum_{n=0}^{N} |a_n|^2 r^{2n}, \tag{7}$$

where $s_1 < r < r_0$. Let

$$\sum_{n=0}^{N} |a_n| z^n = \sum_{n=0}^{\infty} |a'_n| z^n = \varphi(z), \quad a'_n = 0 (n > N);$$

$$h_n(z) = \sum_{k=1}^{\infty} \gamma_k^{(n)} z^k, \quad h(z) = \sum_{k=1}^{\infty} \gamma_k z^k,$$

where $|\gamma_k^{(n)}| \leqq \gamma_k$ for $k = 1, 2, \ldots$; $n = 0, 1, 2, \ldots$.

The left side of (7) is easily reduced to the form

$$\sum_{m=1}^{\infty} r^{2m} \left|\sum_{n=0}^{m-1} a'_n \gamma_{m-n}^{(n)}\right|^2. \tag{8}$$

The expression inside the absolute value signs does not exceed in absolute value the coefficient of z^m in the power series of $\varphi(z)h(z)$, and consequently does not exceed

$$\left|\frac{1}{2\pi i}\int_{|z|=\rho} \frac{h(z)\varphi(z)}{z^{m+1}}\, dz\right|, \quad r < \rho < r_0;$$

and the square of its absolute value does not exceed

$$\frac{K_\rho^2}{2\pi\rho^{2m}}\int_0^{2\pi} |\varphi(re^{i\theta})|^2 d\theta = \frac{K_\rho^2}{\rho^{2m}} \sum_{n=0}^{m-1} |a'_n|^2 \rho^{2n}.$$

Substitution in (8), followed by some algebraic reductions, yields

$$\frac{K_\rho^2 r^2}{\rho^2 - r^2} \sum_{n=0}^{N} |a_n|^2 r^{2n}$$

as an upper bound for (8), and thus (7) is true with $\lambda < 1$ if ρ is chosen so that

$$r < \rho(K_\rho^2 + 1)^{-1/2}.$$

A particular set of functions coming under the hypotheses of Theorem 3 is the set $\{z^n e^{\alpha_n z}\}$, where the α_n are complex numbers such that $|\alpha_n| \leqq 1$. These functions are of interest because, if every function analytic in the circle $|z| < R$ can be expanded in a series of them, the following theorem can be deduced.

THEOREM 4. *If $F(z)$ is an entire function such that*[4]

$$\limsup_{r \to \infty} \frac{1}{r} \log M(r) < R,$$

and $F(z)$ and each of its derivatives have a zero inside or on the unit circle, then $F(z) \equiv 0$.

Theorem 4 has been established by Takenaka[9] with $R = \log 2 = 0.693+$. It has been conjectured[9] that it is true with $R = \pi/4 = 0.785+$ [which would be "best possible:" example, $F(z) = \sin(\pi z/4) - \cos(\pi z/4)$]; the conjecture has been verified by I. J. Schoenberg[10] when all the zeros are required to lie on the real axis.

For the functions $h_n(z) = z^n(e^{\alpha_n z} - 1)$, (8) does not exceed

$$\sum_{m=1}^{\infty} r^{2m} \left\{ \sum_{n=0}^{m-1} \frac{|a'_n|}{(m-n)!} \right\}^2;$$

by retracing the steps which led to (8), one sees that the left side of (7) does not exceed

$$\frac{1}{2\pi} \int_0^{2\pi} \left| (e^z - 1) \sum_{n=0}^{N} |a_n| r^n e^{in\theta} \right|^2 d\theta.$$

This yields at once the expansion theorem with $R = \log 2$ (it was known previously, I believe, only for $R = 1/e$), and so establishes Takenaka's result for Theorem 4.[11] It seems likely that still more careful estimates will yield the value $R = \pi/4$.

Detailed proofs, with extensions of the special theorems and other applications of Theorem 1, will be published elsewhere.

[1] The terminology is that of S. Banach's book, *Théorie des opérations linéaires*, 1932, except that Banach's real linear spaces are replaced by complex linear spaces.

[2] These PROCEEDINGS, **3**, 656–659 (1917).

[3] *C. R. Acad. Sci. Paris*, **163**, 942–945 (1917).

[4] *Trans. Amer. Math. Soc.*, **31**, 53–57 (1929). Other references will be found in this paper.

[5] *Fourier Transforms in the Complex Domain*, 100 (1934).

[6] *Proc. Phys.-Math. Soc. Japan* (3), **13**, 111–132 (1931).

[7] See G. S. Ketchum, *Trans. Amer. Math. Soc.*, **40**, 208–224 (1936), where further references are given.

[8] $M(r)$ denotes the maximum of $|F(z)|$ for $|z| = r$.

[9] See J. M. Whittaker, *Interpolatory Function Theory*, 44–45 (1935).

[10] *Trans. Amer. Math. Soc.*, **40**, 12–23 (1936).

[11] The method indicated here can be developed further: if (6) of Theorem 3 is replaced by $|h(re^{i\theta})| \leqq K_r$, the expansion can be shown to be possible in any subcircle of $|z| < s$ in which $K_r < 1$; Takenaka's theorem can easily be deduced from this result.

SOME COMBINATORIAL PROPERTIES OF COMPLEXES

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1. *Introduction.*—The purpose of this note is twofold. First, we give a definition of the "dual" of a cell in a larger cell, and use this to define products in a complex.[1] Second, we discuss "locally isomorphic" complexes, and products in such complexes. The results are needed in the following note.

2. *Dual Systems in a Complex.*—For each pair of oriented cells σ^r, σ^{r+s} in a geometric complex K, let

$$D(\sigma^r, \sigma^{r+s}) = \text{dual of } \sigma^r \text{ in } \sigma^{r+s}$$

be a singular s-chain whose cells are in σ^{r+s}, and each cell of the closure of which is in either σ^{r+s} or σ^r. Let it be 0 if σ^r and σ^{r+s} are not incident. Set

$$D(\Sigma\alpha_i\sigma_i^r, \Sigma\beta_j\sigma_j^{r+s}) = \Sigma\alpha_i\beta_j D(\sigma_i^r, \sigma_j^{r+s}).$$

If these satisfy the two conditions

$$D(x_i, x_i) = x_i \qquad \text{(for each vertex } x_i\text{)},$$

$$\partial D(\sigma^r, \sigma^{r+s}) = (-1)^{r+1}[D(\delta\sigma^r, \sigma^{r+s}) - D(\sigma^r, \partial\sigma^{r+s})],$$

we say these form a *system dual to* K.

3. *Intersections in a Cell.*—Let E'^n be an oriented half-plane bounded by E^{n-1}: $\partial E'^n = E^{n-1}$. Let

$$\sigma^r = x_{\lambda_0}\ldots x_{\lambda_r},\ \sigma^s = x_{\mu_0}\ldots x_{\mu_s},\ (r + s = n),$$

be simplexes in $E'^n + E^{n-1}$. With a simple definition of *general position relative to* E^{n-1}, we may define their Kronecker index $\{\sigma^r, \sigma^s, E'^n\}$ as usual;

we omit anything in E^{n-1}. (The cells may be singular, if any one is wholly in E'^n or E^{n-1}.) Let their index in any E^* be 0 if they are not both in E^*. Then

$$\{A^r, \partial B^{r+s}, E'^n\} - (-1)^r\{\partial A^r, B^s, E'^n\} + \{\partial A^r, \partial B^s, \partial E'^n\} = 0.$$

Applying the same theory to a cell and its faces gives the same formula with E'^n replaced by a cell or chain C^n.

4. *Products in Terms of Intersections.*—Say two systems D, D' dual to K are in *general position* if their sets of singular cells are in general position in each σ^r relative to $\partial\sigma^r$.

If this holds, and we set

$$(\sigma^r \smile \sigma^s)\cdot\sigma^{r+s} = \{D(\sigma^r, \sigma^{r+s}), D'(\sigma^s, \sigma^{r+s}), \sigma^{r+s}\},$$

then the $\smile$ and corresponding $\frown$ products satisfy the conditions in PC.

5. *Special Products.*—The products of PC, §6, may be obtained as follows: Let K be simplicial, with ordered vertices. Take $0 < a < 1$, and set

$$y_{\lambda_0} = x_{\lambda_0}, \quad y_{\lambda_0\ldots\lambda_r} = (1-a)x_{\lambda_0} + ay_{\lambda_1\ldots\lambda_r}\ (\lambda_0 < \ldots < \lambda_r).$$

$y_{\lambda_0\ldots\lambda_r}$ is interior to $x_{\lambda_0}\ldots x_{\lambda_r}$. Let

$$[\lambda'_0 \ldots \lambda'_r] = +1 \text{ or } -1$$

according as $(\lambda'_0,\ldots,\lambda'_r)$ is an even or odd permutation of the natural order of these numbers. If $(\lambda'_0,\ldots,\lambda'_r)$ is a permutation of the natural order $\lambda_0 < \ldots < \lambda_r$, we use

$$x_{\lambda_0} \ldots x_{\lambda_r} = [\lambda'_0\ldots\lambda'_r]x_{\lambda'_0}\ldots x_{\lambda'_r},\ y_{\lambda'_0}\cdots{}_{\lambda'_r} = y_{\lambda_0}\cdots{}_{\lambda_r}.$$

Now take any cells

$$\sigma^r = x_{\lambda_0}\ldots x_{\lambda_r},\ \sigma^{r+s} = [\lambda_0\ldots\lambda_r\ldots\lambda_{r+s}]x_{\lambda_0}\ldots x_{\lambda_r}\ldots x_{\lambda_{r+s}},\ \lambda_0 < \ldots < \lambda_r,\ \lambda_{r+1} < \ldots < \lambda_{r+s},$$

and set

$$D_a(\sigma^r, \sigma^{r+s}) = \sum_{(\mu)}[\lambda_0\ldots\lambda_r\mu_1\ldots\mu_s]y_{\lambda_0\ldots\lambda_r}y_{\lambda_0\cdots\lambda_r\mu_1}\cdots y_{\lambda_0\ldots\lambda_r\mu_1\ldots\mu_s},$$

summed over all permutations $(\mu_1,\ldots,\mu_s)$ of $(\lambda_{r+1},\ldots,\lambda_{r+s})$. Then *this is a system dual to* K. *If* $0 < a' < a < 1$, *then the systems* D_a, $D_{a'}$ *are in general position, and the definition in* §4 *gives the products of* PC, §6. (This is easy to verify for $r + s \leqq 2$.) If we replace a' by 0 and a by 1, we find interesting combinatorial formulas.

6. *A New Particular $\smile$ Product.*—Writing $A^r = \Sigma a_{\lambda_0 \ldots \lambda_r} x_{\lambda_0} \ldots x_{\lambda_r}$, etc., then $A^r \smile B^s = C^{r+s}$ may be defined as follows, for various r and s:

$$c_\lambda = a_\lambda b_\lambda, \quad c_{\lambda_0 \ldots \lambda_r} = a_{\lambda_0 \ldots \lambda_r} b_{\lambda_0}, \quad c_{\lambda_0 \ldots \lambda_s} = a_{\lambda_1} b_{\lambda_0 \lambda_1 \ldots \lambda_s}, \quad c_{\lambda_0 \ldots \lambda_{r+s}} = (-1)^{r+1} a_{\lambda_1 \ldots \lambda_{r+1}} (b_{\lambda_1 \lambda_{r+1} \ldots \lambda_{r+s}} - b_{\lambda_0 \lambda_{r+1} \ldots \lambda_{r+s}}).$$

This product is used in the following note, §6.

7. *Locally Isomorphic Complexes.*—Let K be an abstract complex. In place of (K_4) of PC, we assume: (*a*) Each σ^1 is incident with exactly two vertices; $\partial\sigma^1 = \pm a \pm b$. (*b*) Each $\partial\sigma^r (r>1)$ is combinatorially connected (through paths with non-zero incidence numbers). (*c*) 0- and 1-cycles mod 2 in any closed cell $\bar{\sigma}$ bound in $\bar{\sigma}$, if the 0-cycles have an even number of vertices. Two complexes K, K' are *locally isomorphic* if there is a (1–1) correspondence ϕ between their non-oriented cells which preserves incidences, and such that for each σ, $\bar{\sigma}$ and $\phi\bar{\sigma}$ are isomorphic complexes. Examples:

$$\begin{aligned} K: &\quad \partial(ab) = b - a, \ \partial(ac) = c - a, \ \partial(bc) = \quad c - b, \\ K': &\quad \partial'(ab) = b - a, \ \partial'(ac) = c - a, \ \partial'(bc) = -c - b. \end{aligned}$$

If $\sigma = bc$, then reorienting c to c' in K' gives $\partial'(bc') = \partial(bc)$. K' has 1-dimensional torsion.

Note that chains mod 2 may be transferred to locally isomorphic complexes at will.

A *unit* 0-*chain* I is a sum $\Sigma(\pm x_i)$ of all oriented vertices. Let a_2 be the integer a mod 2; let $A|_2 = (A)_2$ be the chain A reduced mod 2.

For any I, $W = (\frac{1}{2}\delta I)_2$ exists and is a cocycle; its cohomology class $\mathbf{W}$ is independent of the I chosen. If K and K' are locally isomorphic, then they are isomorphic if and only if $\mathbf{W} = \mathbf{W}'$.

K is *normal* if $\mathbf{W} = 0$.[2] Any geometric complex is normal. Given any normal K and any 1-cocycle X mod 2 in K, there is a locally equivalent K' with $W' = X$.

8. *Particular Orientations in Simplicial Complexes.*—Set

$$\rho(0_2) = 1, \ \rho(1_2) = -1, \ \rho(\Sigma\alpha_i\sigma_i) = \Sigma\rho(\alpha_i)\sigma_i.$$

Let σ_i^r denote oriented cells in either of the locally isomorphic complexes K, K', K being normal; the incidence numbers may differ in sign. Define new incidences in K' as follows. If σ^r, σ^{r+1} have the same first vertex, let $[\sigma^r : \sigma^{r+1}]' = [\sigma^r : \sigma^{r+1}]$. If their first vertices x_i, x_j are different, set

$$[\sigma^r : \sigma^{r+1}]' = \rho(W' \cdot (x_i x_j))[\sigma^r : \sigma^{r+1}].$$

The K' thus constructed is isomorphic with the given one. (The cells may be reoriented.) Thus, in the example, we may use $W' = (bc)_2$, $\rho(W' \cdot bc) =$

$\rho(1_2) = -1$, and $[c:bc]' = -[c:bc] = -1$. We may also write, with the $\frown$ product of PC, §6,

$$[\sigma^r : \sigma^{r+1}] = \rho[W' \cdot (\sigma^r \frown [\sigma^{r+1}])]\,[\sigma^r : \sigma^{r+1}].$$

9. *Products in General Complexes.*—The theory in PC holds only in normal complexes. It may be extended as follows. Let K', K'', K''' be locally isomorphic complexes, with

$$\mathbf{W}' + \mathbf{W}'' + \mathbf{W}''' = 0.$$

Then we may define products of the form

$$X' \smile Y'' = Z''', \qquad X'' \frown A''' = B',$$

as follows. Orient the vertices x_i', etc., so that if $I' = \Sigma x_i'$, etc., then

$$\left(\frac{1}{2}\delta I'\right)_2 + \left(\frac{1}{2}\delta I''\right)_2 + \left(\frac{1}{2}\delta I'''\right)_2 = 0.$$

Let K be the corresponding normal complex, in which $\frown_0$ is defined. Take any face σ_1 of any σ_2; choose a vertex x in σ_1. Let Φ', etc., be isomorphisms of $\bar{\sigma}_2$ into $\bar{\sigma}_2'$, etc., such that $\Phi'(x) = x'$, etc. Then set

$$\sigma_1'' \frown \sigma_2''' = \Phi'[(\Phi'')^{-1}(\sigma_1'') \frown_0 (\Phi''')^{-1}(\sigma_2''')].$$

If K is simplicial, with ordered vertices, we may use, if $\lambda_0 < \ldots < \lambda_p < \ldots < \lambda_{p+q}$,

$$(x_{\lambda_p} \ldots x_{\lambda_{p+q}})'' \frown (x_{\lambda_0} \ldots x_{\lambda_p} \ldots x_{\lambda_{p+q}})''' = \rho[W''(x_{\lambda_0} x_{\lambda_p})]\,(x_{\lambda_0} \ldots x_{\lambda_p})'.$$

The products are unique just as in PC.

10. *Subdivisions; Chain-Mappings.*—Any complex K', in which each $\bar{\sigma}$ has the homology groups of a closed simplex, is locally isomorphic with a similar normal complex K; the latter has a "regular" subdivision, with geometric meaning. The theory of chain-mappings[3] carries over to the present case. Products are preserved under subdivision and chain-mappings. If K_1' is the "regular" subdivision of K', then $W' = Sd'W_1'$.

11. *Coboundary Operators.*—Set $\omega(0_2) = 0$, $\omega(1_2) = -1$ (we could use $+1$). Set $\omega(\Sigma\alpha_i\sigma_i^r) = \Sigma\omega(\alpha_i)\sigma_i^r$ ($\alpha_i = 0_2$ or 1_2). Then for any p-I_2-cocycle X, $\frac{1}{2}\delta\omega X$ is a $(p+1)$-I_0-cocycle whose cohomology class depends only on that of X. The dual operation $\frac{1}{2}\partial\omega A$ is well known.

If K' and K'' are locally isomorphic, and we let σ_i^r denote cells in either, then $\frac{1}{2}(\delta''A' - \delta'A')$ is defined for all chains A' in K'; as a chain in K'', if $\delta'A' = 0$, it is a cocycle whose class depends only on that of A'. If $\delta'A' = 0$, we may write it as $\frac{1}{2}\delta''\omega(A\,|_2)$.

Corresponding to the pair K', K'' is a characteristic 2-I_0-cocycle in the corresponding complex K''',

$$X''' = \frac{1}{4}\delta''\delta' I' \qquad (I' = \Sigma x_i').$$

If $\mathfrak{B}'$, $\mathfrak{B}''$ are 0-sphere-bundles with these as classes, then their product has the 2-class X''' (see the following note). Similarly, to K', K'', K''' corresponds $\frac{1}{8}\delta'''\delta''\delta' I'$, etc.

12. *Relation to Products.*—Using K', K'', K''' again, and $\omega A = A$ if A has integer coefficients, set

$$X' \smile Y'' = \omega X' \frown \omega Y''.$$

This has topological significance if each chain either is integral or equals W' or W''. To show this, we use, if σ_2 is a p-face of $\sigma_1 = \sigma_1^{p+1}$, and x_2, x_1 are their first vertices,

$$\frac{1}{2}(\delta'''\sigma_2 - \delta''\sigma_2)\cdot\sigma_1 = \rho[W''(x_1x_2)]\,\omega W'(x_1x_2).$$

We find

$$W' \smile Y'' = \frac{1}{2}(\delta'''Y'' - \delta''Y''),$$

$$W' \smile W'' = \frac{1}{2}(\delta'''\omega W'' - \delta''\omega W'') = \text{the } X''' \text{ above.}$$

13. *On Non-Orientable Manifolds.*[4]—Let K be a subdivision of the non-orientable manifold M. For each oriented σ, let ϵ be an orientation of the part of M about σ; set

$$\sigma' = (\sigma, \epsilon) = (-\sigma, -\epsilon),\ -\sigma' = (-\sigma, \epsilon) = (\sigma, -\epsilon),\ [\sigma_1' : \sigma_2'] = [\sigma_1 : \sigma_2] \text{ or } -[\sigma_1 : \sigma_2]$$

according as the orientations ϵ_1 and ϵ_2 agree or disagree. The new complex K' is locally isomorphic with K; its class has the property that any 1-circuit A^1 in K preserves orientation if and only if $A^1 \cdot W' = 0_2$. The Poincaré duality theorem (compare PC, §18) says:

$${}^p\mathbf{H}_G(K) \approx {}^{n-p}\mathbf{H}^G(K'), \qquad {}^p\mathbf{H}_G(K') \approx {}^{n-p}\mathbf{H}^G(K).$$

Note that in K', the sum of the n-cells forms a cycle with integral coefficients.

[1] We recall the notations ∂ for boundary, δ for coboundary; I_μ = integers mod μ, I_0 = integers. An r-I_μ-chain is an r-chain with coefficients in I_μ. The reference PC will mean Whitney, "On Products in a Complex," *Ann. Math.*, **39**, 397–432 (1938). We note some corrections to this paper: In footnote 9, 3-sphere should read 3-complex.

In Theorem 12, we mean the (uniquely determined) products of 11 (a). In place of the following paragraphs, read: As knowing all $v_i^p \cdot u^p$ for all i uniquely determines u^p, and the correct $\frown$ satisfies (11.8), we have found the correct $\frown$. (NOTE: Theorem 11 is not used.) The end of the first paragraph in 12 should read: Find $u^p \smile u^q$ over I_0, then over R_1, ($p \leqq 2$), then for all p, by (5.12); then find $\frown$ over I_0, by Theorem 12. (We must know (11.15).) In Theorem 13, (a), add: $\phi\tau^p = 0$ if $O(\tau^p)$ is acyclic. Relation (14.9) follows directly from (14.4). After (25.7), add: $I^p \smile I^q = I^{p+q}$.

[2] *Augmentable* in Tucker, *Ann. Math.*, **34**, 191–243 (1933).

[3] See Tucker, these PROCEEDINGS, **25**, 371–374 (July, 1939). The theory was developed independently by S. Lefschetz and myself.

[4] This case of the theory is due to de Rham; see *Comm. Math. Helv.*, **4**, 151–157 (1933).

ON THE THEORY OF SPHERE-BUNDLES

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1. *Introduction.*—We give here a brief sketch of some new results in the theory of sphere-bundles;[1] in particular, further properties of the characteristic classes, a duality theorem, theorems on tangent and normal bundles to a manifold, and some examples. The results will be published later in book form.

2. *Fibre-Bundles.*—Let S_0 be a space, and G, a group of homeomorphisms of S_0 into itself. Then over any space K, the *base space*, with neighborhoods U_i, we may define a *fibre-bundle* $\mathfrak{B}(K)$ as follows. For $p \in U_i$, let $\xi_i(p)$ be a homeomorphism of S_0 into a set of points $S(p)$. Let $S(p) \cdot S(q) = 0$ if $p \neq q$. Let $\mathfrak{S}(K)$ be the space of all points on all $S(p)$, the *total space* (gefaserte Raum). Let $\xi_i(p, q)$ be the image of q in S_0 under $\xi_i(p)$. For $p \in U_i \cdot U_j$, set $\xi_{ij}(p, q) = \xi_i^{-1}(p, \xi_j(p, q))$; assume $\xi_{ij}(p) \in G$ for each p, and that it varies continuously with p. Then a topology is easily defined in $\mathfrak{S}$. The part $\mathfrak{S}(U_i)$ of $\mathfrak{S}$ over U_i is a product $U_i \times S_0$.

If S_0 is a set of μ points, and G is the group of permutations, we obtain the covering spaces of K with μ sheets. If S_0 is a subgroup of a continuous group R_0, and $G = S_0$, then the left (or right) cosets of S_0 form a space K (factor group if S_0 is normal); the total space is R_0. If $S_0 = S_0^\nu$ is a ν-sphere, and $G = G^{\nu+1}$, the orthogonal group, we have a *sphere-bundle*. If S_0 is a vector space, and G, the affine or orthogonal group, an equivalent theory is obtained.

3. *Particular Coördinate Systems.*—We use this section in the proof of the duality theorem. Let K be a complex with ordered vertices. We may use ξ_σ, defined over closed cells σ. (See TP.) We may choose them so $\xi_\sigma = \xi_{\sigma'}$ if σ and σ' have the same first vertex. Let P^ν be a small closed

region in K surrounding all $D_a(\sigma^r, \sigma')$, σ' of any dimension (see I, §2); set $Q^r = \overline{K - P^r}$. Then if K^r is the r-dimensional part of K, and $e_1, \ldots, e_{\nu+1}$ are the unit points of S_0^ν,

$$\xi_{\sigma,\sigma'}(p,e_i) = e_i \qquad (p \in Q^k \cdot \sigma \cdot \sigma', i = 1, \ldots, \nu - k).$$

Note that $\phi_i(p) = \xi_{\sigma^{r-1}}(p, e_i)$ ($p \in$ any σ^{r-1}, $i = 1, \ldots, \nu - r + 2$) defines orthogonal projections of K^{r-1} into $\mathfrak{S}$.

4. *Characteristic Classes.*—Choose $\phi_1, \ldots, \phi_{\nu-r+2}$ as above over K^{r-1}; then for each σ^r, studying these on $\partial\sigma^r$ gives $W^r \cdot \sigma^r$, which is an integer mod 2 if $r = 1$ or $r \leqq \nu$ is even, and an integer otherwise. W^r is a cocycle whose class $\mathbf{W}^r$ is an invariant of $\mathfrak{B}$; the $\mathbf{W}^r$ characterize $\mathfrak{B}$ if $\nu \leqq 1$ or dim $(K) \leqq 3$ (see TP). We may use a general type of subdivision of the polyhedron K in defining the $\mathbf{W}^r$.

If $\mathfrak{B}$ is not orientable (TP, §4), and K' is obtained from K by replacing $[\sigma^r : \sigma^{r+1}]$ by $-[\sigma^r : \sigma^{r+1}]$ when ξ_{σ^r} and $\xi_{\sigma^{r+1}}$ give opposite orientations to the $S(p)$ ($p \in \sigma^r$) (see TP, p. 793, footnote), then K' is locally isomorphic with K, and $\mathbf{W}' = \mathbf{W}^1$. We call K' the *complex associated with* $\mathfrak{B}$. The characteristic classes are taken in K'; the theorems above hold still.

If f maps K_1 into K_2, and $\mathfrak{B}_2(K_2)$ is defined, then a bundle $\mathfrak{B}_1(K_1)$ is defined (TP, §8), and $\mathbf{W}_1^r = f'\mathbf{W}_2^r$ (f' = dual of f).

If $\nu = 2$, dim$(K) = 4$, and $\mathbf{W}^1 = 0$, $\mathbf{W}^2 = 0$, then an invariant characterizing $\mathfrak{B}$ is obtained as follows. A triple $\phi = (\phi_1, \phi_2, \phi_3)$ of orthogonal projections of K^3 into $\mathfrak{S}(K^3)$ exists. Let $\Phi(p)$ ($p \in K^3$) map S_0 into $S(p)$ so that $\Phi(p, e_i) = e_i$ ($i = 1, 2, 3$). For each σ^4, set

$$\Psi_{\sigma^4}(p) = \xi_{\sigma^4}^{-1}(p)\Phi(p) \quad (p \in \partial\sigma^4).$$

This maps $\partial\sigma^4$ into the orthogonal group G^3; as G^3 is homeomorphic with projective 3-space P^3, this defines an integer $D^4 \cdot \sigma^4$, the degree of Ψ_{σ^4}. D^4 is a cocycle. If we identify two cocycles if they are cohomologous, or differ by a cocycle of the form $X^1 \smile X^1 \smile X^1 \smile X^1$ (X^1 a 1-cocycle), the class determined is the invariant.

The classes W^{2r+1} are determined from the others as follows (see I, §11):

$$W^{2r+1} = \tfrac{1}{2}\partial\omega W^{2r} \qquad (\text{if } \nu \geqq 2r).$$

5. *On Mappings into* $G^{\nu+1}$.—In the theorem just stated, and in the duality theorem, we need the following (and other more complicated) theorems. (*a*) Let f map σ^ν into $G^{\nu+1}$ so that if $\phi(p) = f(p,e_1)$, then $\phi(p) = e_1$ in $\partial\sigma^\nu$; let ϕ be of degree α. Let ψ map σ^ν into S_0^ν with the degree β, and let $\psi(p) = e_1 (p \in \partial\sigma^\nu)$. Then $\theta(p) = f(p, \psi(p))$ is of degree $\alpha + \beta$. (*b*) Take f as before; then $\phi'(p) = f^{-1}(p, e_1)$ (the point of S_0 mapped into e_1 by $f(p)$) is of degree $-\alpha$. (Use (*a*).) (*c*) Let f map σ^ν into $G^{\nu+1}$, let ϕ map $\partial\sigma^\nu$ into $S_0^{\nu-1}$ with the degree α, and suppose $f(p, \phi(p)) = e_1$ ($p \in$

$\partial\sigma^r$). Then $\psi(p) = f(p, e^{r+1})$ maps $\partial\sigma^r$ into the S_1^{r-1} orthogonal to e_1 with a degree $\equiv \alpha \pmod 2$.

6. *The Duality Theorem.*—Given bundles $\mathfrak{B}_1^\lambda(K)$ and $\mathfrak{B}_2^\mu(K)$, there is a uniquely determined bundle $\mathfrak{B}_3^\nu(K)$, their *product* ($\nu = \lambda + \mu + 1$; see TP, p. 796); thus if $M^m \subset M^n$, the tangent times the normal bundle gives the part of the tangent bundle of M^n over M^m. The formula for the characteristic classes of $\mathfrak{B}_3^r$ is

$$\mathbf{W}_3^r = \sum_i \mathbf{W}_1^i \smile \mathbf{W}_2^{r-i}, \text{ reducing mod 2 if necessary.}$$

(See §4 and I, §12; we use W^0 = sum of vertices.) The proof is very difficult if $r \geqq 4$. We use the special $\xi_{1,\sigma}$ of §3 in $\mathfrak{B}_1$, and $\xi_{2,\sigma}$ in $\mathfrak{B}_2$, with a replaced by $a' < a$, so the P_1 and P_2 will be in "general position" (see I, §5). The projections into $\mathfrak{S}_3(K^{r-1})$ are defined successively over Q_1^0, Q_1^1, ... For each σ^r, they are now deformed in $\partial\sigma^r$ into a simpler position, except in each $A^i = P_1^i \cdot P_2^{r-1-i} \cdot \sigma^{r-1}$ (σ^{r-1} = face of σ^r opposite first vertex of σ^r). The terms shown come from the A^i, two coming from A^{r-1}. The results of §5 and the products of I, §6, are needed.

REMARK. We do not know whether or not the individual terms $\mathbf{W}_1^i \smile \mathbf{W}_2^{r-i}$ have topologicial significance.

Reducing everything mod 2, write, for any $\mathfrak{B}$, the formal power series

$$\mathbf{W} = \sum_i \mathbf{W}^i t^i, \quad \overline{\mathbf{W}} = 1/\mathbf{W} = \sum_i \overline{\mathbf{W}}^i t^i;$$

then

$$\overline{\mathbf{W}}^0 = I|_2, \overline{\mathbf{W}}^1 = \mathbf{W}^1, \overline{\mathbf{W}}^2 = \mathbf{W}^2 + \mathbf{W}^1 \smile \mathbf{W}^1,$$

etc. The duality theorem gives then, as $\mathbf{W}_N = \mathbf{W}/\mathbf{W}_T$, etc.,

$$\mathbf{W}_N{}^r = \sum \mathbf{W}^i \smile \overline{\mathbf{W}}_T{}^{r-i}, \quad \overline{\mathbf{W}}^r = \sum \overline{\mathbf{W}}_T{}^i \smile \overline{\mathbf{W}}_N{}^{r-i}, \text{ etc. (mod 2).}$$

7. *Tangent Bundles.*—Let K be a simplicial subdivision of the manifold M^n, with ordered vertices. Each p in K may be written uniquely as $p = \sum \eta_{\lambda_i}(p) x_{\lambda_i}$, if $p \in x_{\lambda_0} \cdots x_{\lambda_r}$. Define

$$v_k(p) = \sum_{\lambda_0 < \cdots < \lambda_k} \eta_{\lambda_0}(p) \cdots \eta_{\lambda_k}(p)(x_{\lambda_k} - x_{\lambda_{k-1}}) \quad (k = 1, 2, \ldots).$$

These are continuous in K, and the first r are independent except in K^{r-1} (any r). If K^* is the usual complex dual to K, these may be used to define W^r, a cocycle in K^*. Its dual is a *characteristic cycle* C^{n-r} in K', the complex associated with $\mathfrak{B}$ (which was studied in I, §13). Note that $C^{n-(2r+1)} = \frac{1}{2}\partial\omega C^{n-2r}$. The value of $C^s \cdot \sigma^s$ ($s = n - r$, $\sigma^s = x_{\lambda_0} \cdots x_{\lambda_s}$) is as fol-

lows. Let K_1 be the subcomplex of the closed star of σ^s containing all vertices x_i with

$$\lambda_s > i > \lambda_{s-1} \text{ or } \lambda_{s-2} > i > \lambda_{s-3} \text{ or } \ldots.$$

(This includes all vertices below λ_0 if s is even.) Then $C^s \cdot \sigma^s = 1 - \chi(K_1)$ (χ = Euler-Poincaré characteristic), or this mod 2.

From this we prove: If K is the first derived of a simplicial subdivision of M, then C^s is the sum of all s-simplexes of K (properly oriented if integer coefficients are used).[2]

In the proofs of the following theorems, we study the classes over submanifolds of the given manifold, and use the duality theorem and results from §8. For $M^m \subset M^n$, let $\mathbf{W}^r$ mean the part of $\mathbf{W}_T{}^r(M^n)$ in M^m.

If a closed M^m can be imbedded in E^n (with or without singularities), then

$$0 = \mathbf{W}_N{}^{n-m}\big|_2 = \sum \mathbf{W}^i \smile \overline{\mathbf{W}}_T{}^{n-m-i} = \overline{W}_T{}^{n-m}.$$

Hence $\overline{W}_T{}^m = 0$ always. This gives, if $(X)^2 = X \smile X$, etc.,

closed M^2: $\mathbf{W}^2\big|_2 = (\mathbf{W}^1)^2$; closed M^3: $(\mathbf{W}^1)^3 = 0$;

closed M^4: $\mathbf{W}^4\big|_2 + (\mathbf{W}^2)^2 + \mathbf{W}^2 \smile (\mathbf{W}^1)^2 + (\mathbf{W}^1)^4 = 0$; etc.

In any M^2, for any 1-I_2-cocycle X^1, $X^1 \smile X^1 \backsim X^1 \smile W^1$. In any M^3, $\mathbf{W}^2 = \mathbf{W}^1 \smile \mathbf{W}^1$; hence (Stiefel) for orientable M^3 (closed or not), the tangent bundle is simple. In any orientable M^4, $\mathbf{W}^3 = 0$. (The proof uses facts from §10.) For any 2-I_2-cocycle X^2 in any M^4, $X^2 \smile X^2 \backsim X^2 \smile \overline{W}^2$. For any orientable M^4 in an orientable M^r, $\overline{W}_T{}^4 = W_N{}^4$.

8. *Normal Bundles.*—For any M^m in any M^n, $\mathbf{W}_N{}^{n-m}$ is the intersection of M^m with itself in M^n, which is a cohomology class of the complex associated with the normal bundle; if M^m is closed and $M^n = E^n$, then $\mathbf{W}_N{}^{n-m}\big|_2 = 0$, and if also M^m is orientable, then $\mathbf{W}_N{}^{n-m} = 0$. Compare PC, §20, and TP, p. 795.

If M^m is mapped regularly into M^n, but with singularities, we may deform M^m slightly into M'^m, and consider the intersections of the σ' with a neighborhood of σ' in M; then W^{n-m} is the *local intersection* thus defined. For a closed orientable $M^m \subset E^n$, the *intersection* vanishes, so that the *distant intersection* equals the local. This holds mod 2 in the non-orientable case. If $n = 2m$, and the singularities are isolated points, the distant intersection is of course 0 (mod 2); if m is odd, it vanishes, because $\{\sigma^m, \sigma'^m\} = -\{\sigma'^m, \sigma^m\}$.

Take an orientable $M^m \subset E^n$. Then the normal bundle is simple if $m = 1$ or 2, or $n = m + 1$ or $m + 2$, or $m = 3$ and M is closed, or M is a cell. This holds if M^m is merely mapped regularly, provided that if $n = m + 2$, then m is odd, and we omit $m = 2$ if M is closed.

9. *Examples.*—Consider a cylinder, the product $T^3 = T^1 \times T^2$ of a segment and a disc. Let P^2 be one end, let $S_2{}^1$ be a segment crossing P^2,

let Q^2 be a rectangle cutting through T^3 and ending on $S_2{}^1$, and let $S_1{}^1$ be the center of Q^2, ending at p_0, the center of S_2; rather, let these be the sets after the identifications below. (1) Join the ends of T^3, and shrink ∂T^2 to a point; this forms $M_1{}^3 = S^1 \times S^2$, with a simple tangent bundle. (2) Join the ends, and identify opposite points of ∂T^2; then characteristic cycles are (mod 2) $C^2 = Q^2$, $C^1 = S_1{}^1$; $\therefore \mathbf{W}^2 \neq 0$. (3) Join the ends, reflecting one so that P^2 is joined to itself with orientation reversed, and shrink ∂T^2 to a point. We obtain $M_2{}^3$, with $\mathbf{W}^2 = 0$. (4) Join the ends as in (3), and identify opposite points of ∂T^2, forming $M_4{}^3$. Now P^2 is a projective plane, and Q^2 is a Klein bottle. Intersections are (mod 2)

$$\{P^2, P^2\} \sim 0,\ \{Q^2, P^2\} \sim S_2{}^1,\ \{Q^2, Q^2\} \sim S_1{}^1 + S_2{}^1;$$
$$\{P^2, S_2{}^1\} \sim 0,\ \{P^2, S_1{}^1\} \sim p_0,\ \{Q^2, S_2{}^1\} \sim p_0,\ \{Q^2, S_1{}^1\} \sim 0;$$

characteristic classes are

$$C^2 \sim (P^2 + Q^2)_2,\ C^1 \sim (S_1{}^1 + S_2{}^1)_2.$$

Define $\mathfrak{B}_1{}^1(M_4{}^3)$, with $C_1{}^2 \sim P^2$, $C_1{}^1 \sim 0$, and $\mathfrak{B}_2{}^1(M_4{}^3)$, with $C_2{}^2 \sim Q^2$, $C_2{}^1 \sim 0$; let the total spaces be $M_1{}^4$, $M_2{}^4$. We may pretend $M_4{}^3$ is in either (because $C_i{}^1 = 0$). Then

$$C_1{}^3 \sim \mathfrak{S}(Q^2),\ C_1{}^2 \sim \mathfrak{S}(S_1{}^1),\ C_1{}^1 \sim \mathfrak{S}(p_0);\ \ \therefore \overline{C}_1{}^1 \sim 0;$$
$$C_2{}^3 \sim \mathfrak{S}(P^2),\ C_2{}^2 \sim \mathfrak{S}(S_2{}^1),\ C_2{}^1 \sim \mathfrak{S}(p_0);\ \ \therefore \overline{C}_2{}^1 \sim \mathfrak{S}(p_0);$$

hence in $M_2{}^4$, $\mathbf{W}^3 \neq 0$, $\overline{\mathbf{W}}^3 \neq 0$. Hence (see §7) $M_2{}^4$ cannot be imbedded in E^7.

Define $\mathfrak{B}^2(M_3{}^3)$, with $C^2 \sim P^2$, and $C^1 \sim S_1{}^1$. Then we may consider $M_3{}^3 \subset M^5 = \mathfrak{S}(M_3{}^3)$, and prove ($a$) M^5 is closed and orientable, (b) $C^2(M^5) \sim \mathfrak{S}(p)$; hence $\overline{\mathbf{W}}^3 = \mathbf{W}^3 \neq 0$, and M^5 cannot be imbedded in E^8.

We may define $M^8 = \mathfrak{S}(S_0{}^4)$, with $\overline{\mathbf{W}}^4 = \mathbf{W}^4 \neq 0$; hence M^8 cannot be imbedded in E^{12}.

The complex projective plane P^{*4} cannot be imbedded in E^6, as $\mathbf{W}^2 \neq 0$. ($\mathbf{W}^2 \smile \mathbf{W}^2 = \mathbf{W}^4\big|_2$; $W^4 \cdot P^{*4} = \chi(P^{*4}) = 3$.) But it can be in E^7.

For any closed orientable M^4, and any cocycles X^2, X^4 (mod 2) in M^4, we may imbed M^4 in an $M^8 \subset E^{17}$, so that the part over M^4 of the characteristic classes of the normal bundle of $M^8 \subset E^{17}$ are X^2 and X^4. Hence we may make $\mathbf{W}_N{}^3 \neq 0$ and $\mathbf{W}_N{}^4 \neq 0$ also.

If we put a Klein bottle Q^2 in E^3, then the distant and local intersections are equal (mod 2); these are $\mathrm{W}_N{}^1 = \mathrm{W}_T{}^1$; $C_T{}^1 =$ a closed curve in Q^2. Hence the distant intersection, as a cycle, is a certain curve in Q^2, as is clear in the usual immersion of Q^2 in E^3. For P^2 in E^3, we get the "projective line" similarly.

A direct study shows: If $M^2 \subset E^4$ is closed (with or without singularities), and C^2 is the fundamental cycle of the associated complex (integer

coefficients), then $W^2 \cdot C^2 = 2[\chi(M^2) + 2k]$ for some k; any k may be obtained. Hence if $\chi \not\equiv 0 \pmod 2$, then a field of normal vectors never exists. Also one imbedding cannot be deformed into another with a different $W^2 \cdot C^2$.

10. *Homology Groups of Total Spaces, Etc.*—Let K be connected. Given $\mathfrak{B}(K)$, the homology groups satisfy $\mathbf{H}^r(\mathfrak{S}) \approx \mathbf{H}^r(K)$, $r < \nu$. $kS(p) \sim 0$ if and only if for some $A^{\nu+1}$, $A^{\nu+1} \cdot W^{\nu+1} = k$. (We may use $k_\lambda = k \bmod \lambda$, and $A^{\nu+1}|_\lambda$.) For $\mathfrak{B}$ oriented, a $(\nu+1)$-I_μ-cycle A in K is the projection of such a cycle in $\mathfrak{S}$ if and only if $A \cdot W^{\nu+1} = 0_\lambda$. Now $\mathbf{H}^r(\mathfrak{S})$ may be described in terms of $\mathbf{W}^{\nu+1}$ and properties of $kS(p)$. If μ is the smallest integer such that $\mu S(p) \sim 0$, then $\mathbf{H}^r(\mathfrak{S}) \approx I_\mu \oplus \mathbf{H}^r(K)$ if and only if for each λ and each $(\nu+1)$-I_λ-cycle A, $A \cdot W^{\nu+1} \equiv 0 \bmod (\lambda, \mu)$. A mapping f of a complex K' of dimension $\leqq \nu+1$ into K is the projection of a mapping into $\mathfrak{S}$ if and only if $f'\mathbf{W}^{\nu+1} = 0$.

[1] We refer the reader to papers in these PROCEEDINGS, **21**, 464–468 (1935), and in *Bull. Am. Math. Soc.*, **43**, 785–805. We denote the latter by TP, and the preceding note, by I. Sphere-bundles were formerly called "sphere-spaces."

[2] This was a conjecture of Stiefel, *Comm. Math. Helv.*, **8**, 40 (1936).